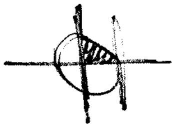
Study Guide #3 (254)

1. $\int_0^4 \int_0^{\sqrt{16-x^2}} \int_0^{10-x-y} dz\,dy\,dx$

$x=4,\ x=0$

$y=0,\ y=\sqrt{16-x^2} \Rightarrow x^2+y^2=16$



$z=0,\ z=10-x-y$ $\Rightarrow x=\sqrt{16-y^2}$

$dz\,dx\,dy \Rightarrow \int_0^4 \int_0^{\sqrt{16-y^2}} \int_0^{10-x-y} dz\,dx\,dy =$

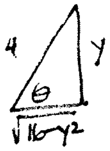
$$\int_0^4 \int_0^{\sqrt{16-y^2}} 10-x-y\ dx\,dy = \int_0^4 10x - \frac{1}{2}x^2 - xy \Big|_0^{\sqrt{16-y^2}} dy =$$

$$\int_0^4 10\sqrt{16-y^2} - \frac{1}{2}(16-y^2) - \sqrt{16-y^2}\cdot y\ dy =$$

$$10\int_0^4 \sqrt{16-y^2}\,dy - \frac{1}{2}\int_0^4 16-y^2\,dy - \int_0^4 y\sqrt{16-y^2}\,dy$$

$y = 4\sin\theta$

$dy = 4\cos\theta\,d\theta$



$10\int 4\cos\theta \cdot 4\cos\theta\,d\theta$

$\frac{160}{2}\int 1+\cos 2\theta\ d\theta$

$80\left[\theta + \frac{1}{2}\sin 2\theta\right]$

$\sin\theta\cos\theta$

$80\left[\arcsin\left(\frac{y}{4}\right) + \frac{y}{4}\cdot\frac{\sqrt{16-y^2}}{4}\right]_0^4$

$80\left[\frac{\pi}{2} + 1(0)\right]$

40π

$-\frac{1}{2}\left[16y - \frac{1}{3}y^3\right]_0^4$

$-\frac{1}{2}\left[64 - \frac{64}{3}\right]$

$-\frac{1}{2}\left[\frac{128}{3}\right]$

$-\frac{64}{3}$

$u = 16-y^2$

$-\frac{1}{2}du = \cancel{-2}y\,dy$

$+\frac{1}{2}\int u^{1/2}\,du$

$\frac{1}{\cancel{2}}\frac{\cancel{2}}{3}u^{3/2}$

$\frac{1}{3}(16-y^2)^{3/2}\Big|_0^4$

$\frac{1}{3}(0)^{3/2} - \frac{1}{3}(16)^{3/2}$

$-\frac{1}{3}\cdot 4^3$

$-\frac{64}{3}$

$-\frac{64}{3} - \frac{64}{3} =$

$\boxed{40\pi - \frac{128}{3}}$

$$\int_0^{\pi/2}\int_0^4\int_0^{10-r\cos\theta-r\sin\theta} r\,dz\,dr\,d\theta =$$

$$\int_0^{\pi/2}\int_0^4 10r - r^2\cos\theta - r^2\sin\theta\,dr\,d\theta$$

$$\int_0^{\pi/2} 5r^2 - \tfrac{1}{3}r^3\cos\theta - \tfrac{1}{3}r^3\sin\theta\Big|_0^4\,d\theta =$$

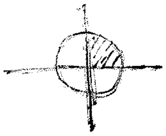
$$\int_0^{\pi/2} 80 - \tfrac{64}{3}\cos\theta - \tfrac{64}{3}\sin\theta\,d\theta$$

$$80\theta - \tfrac{64}{3}\sin\theta + \tfrac{64}{3}\cos\theta\Big|_0^{\pi/2} = 80\cdot\tfrac{\pi}{2} - \tfrac{64}{3}(1) + 0 - 0 + 0 - \tfrac{64}{3}(1) =$$

$$40\pi - \tfrac{128}{3}$$

2. $$\int_0^2\int_0^{\sqrt{4-x^2}}\int_0^{\sqrt{16-x^2-y^2}} x\,dz\,dy\,dx$$

$$\int_0^{\pi/2}\int_0^2\int_0^{\sqrt{16-r^2}} r^2\cos\theta\,dz\,dr\,d\theta$$



$4-x^2=y^2$

$4=x^2+y^2 \Rightarrow r=2$

$$\int_0^{\pi/2}\int_0^2 r^2\sqrt{16-r^2}\cos\theta\,dr\,d\theta$$

cylindrical needs trig sub.

$x^2+y^2\ z^2=16 \quad \rho=4$

top half of hemisphere

1st octant $x\geq 0, y\geq 0, z\geq 0$

$$\int_0^{\pi/2}\int_0^{\pi/2}\int_0^4 \rho\cos\theta\sin\varphi\,\rho^2\sin^2\varphi\,d\rho\,d\theta\,d\varphi =$$

$$\int_0^{\pi/2}\int_0^{\pi/2}\int_0^4 \rho^3\sin^3\varphi\cos\theta\,d\rho\,d\theta\,d\varphi = \int_0^{\pi/2}\int_0^{\pi/2}\tfrac{1}{4}\rho^4\Big|_0^4\sin^3\varphi\cos\theta\,d\theta\,d\varphi =$$

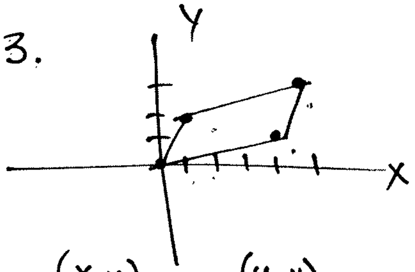
$$64\int_0^{\pi/2}\int_0^{\pi/2}\sin\varphi(1-\cos^2\varphi)\cos\theta\,d\theta\,d\varphi = 64\int_0^{\pi/2}\underset{1}{\cancel{\sin\theta}}\Big|_0^{\pi/2}\sin\varphi(1-\cos^2\varphi)\,d\varphi$$

$u=\cos\varphi$

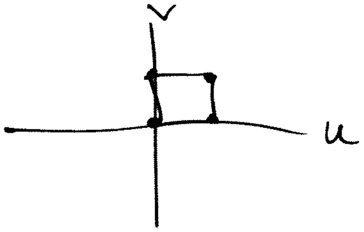
$-du=\sin\varphi$

$$-64\left[\cos\varphi - \tfrac{1}{3}\cos^3\varphi\right]_0^{\pi/2} = -64\left[0-0-1+\tfrac{1}{3}\right] = \boxed{\tfrac{128}{3}}$$

3.



(x,y)		(u,v)
(0,0)	→	(0,0)
(1,2)	→	(0,1)
(5,3)	⇒	(1, 1)
(4,1)	⇒	(1,0)



$4u+v= x \quad u+2v=y$

$$\begin{aligned}-8u-2v&=-2x\\ \underline{u+2v}&\underline{=y}\\ -7u&=y-2x\\ u&=\tfrac{1}{7}(2x-y)\end{aligned}$$

$\frac{1}{7}(2(1)-2)=0$

$\frac{1}{7}(2(5)-3)=1$

$\frac{1}{7}(2(4)-1)=1$

$$\begin{aligned}4u+v&=x\\ \underline{-4u-8v}&\underline{=-4y}\\ -7v&=x-4y\\ v&=\tfrac{1}{7}(4y-x)\end{aligned}$$

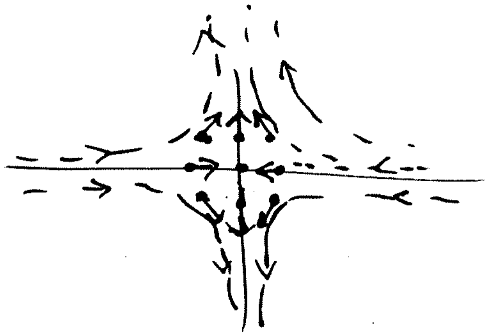
$\frac{1}{7}(4(2)-1)=1$

$\frac{1}{7}(4(3)-5)=1$

$\frac{1}{7}(4(1)-4)=0$

4. $\vec{F}(x,y)=-x\vec{i}+y\hat{j}$

x	y	<x,y>
0	0	<0,0>
1	0	<-1,0>
0	1	<0,1>
-1	0	<1,0>
0	-1	<0,-1>
1	1	<-1,1>
1	-1	<-1,-1>
-1	1	<1,1>
-1	-1	<1,-1>



5. $\vec{F}(x,y) = \dfrac{2x\hat{i}+2y\hat{j}}{(x^2+y^2)^2}$

$\displaystyle\int \frac{2x}{(x^2+y^2)^2}dx = \int u^{-2}\,du = \frac{-1}{u} = \frac{-1}{x^2+y^2}$

$u=x^2+y^2$

$du= 2x\,dx$

$\displaystyle\int \frac{2y}{(x^2+y^2)^2}\,dy = \int u^{-2}\,du = -\frac{1}{u}$

$= -\dfrac{1}{x^2+y^2}$

$u=x^2+y^2$

$du=2y\,dy$

$f(x,y) = \dfrac{-1}{x^2+y^2}+C$

$$\frac{d}{dx}\left[\frac{2y}{(x^2+y^2)^2}\right] = 2y(-2)(x^2+y^2)^{-3}(2x) = \frac{-8xy}{(x^2+y^2)^3}$$

$$\frac{d}{dy}\left[\frac{2x}{(x^2+y^2)^2}\right] = 2x(-2)(x^2+y^2)^{-3}2y = \frac{-8xy}{(x^2+y^2)^3}$$

these match so it is conservative $\ni$ $f(x,y) = \frac{-1}{(x^2+y^2)} + C$ is the potential function.

6. $\vec{F}(x,y,z) = 3x^2y^2z\hat{i} + 2x^3yz\hat{j} + x^3y^2\hat{k}$

$$\nabla \times \vec{F} \quad \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3x^2y^2z & 2x^3yz & x^3y^2 \end{vmatrix} = (2x^3y-2x^3y)\hat{i} - (3x^2y^2-3x^2y^2)\hat{j} + (6x^2yz-6x^2yz)\hat{k}$$

$$= \vec{0}$$

The field is conservative

$$\int 3x^2y^2z\,dx = x^3y^2z + f(y,z)$$

$$\int 2x^3yz\,dy = x^3y^2z + g(x,z)$$

$$\int x^3y^2\,dz = x^3y^2z + h(x,y)$$

$f(x,y,z) = x^3y^2z + C$ is the potential function

7. $\vec{F}(x,y,z) = x^2z\hat{i} - 2xz\hat{j} + yz\hat{k}$

$$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2z & -2xz & yz \end{vmatrix} = (z+2x)\hat{j} - (0-0)\hat{j} + (-2z-0)\hat{k}$$

$$= (z+2x)\hat{i} - 2z\hat{k}$$

8. $\vec{F}(x,y,z) = xe^x\hat{\imath} + ye^y\hat{\jmath}$

$\nabla\cdot F = e^x + xe^x + ye^y + e^y$

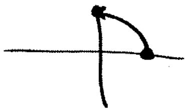
9. $\int_c 4xy\,ds$ $\quad r(t) = t\hat{\imath} + (1-t)\hat{\jmath}$ $\quad r'(t) = 1\hat{\imath} + (-1)\hat{\jmath}$

$x = t \quad y = 1-t$ $\quad \|r'(t)\| = \sqrt{1^2+1^2} = \sqrt{2}$

$\int_0^1 4(t)(1-t)\sqrt{2}\,dt$ $\quad ds = \|r'(t)\|\,dt = \sqrt{2}\,dt$

$4\sqrt{2}\int_0^1 t - t^2\,dt = 4\sqrt{2}\left[\frac{1}{2}t - \frac{1}{3}t^3\right]_0^1 = 4\sqrt{2}\left[\frac{1}{2} - \frac{1}{3}\right] = \frac{4\sqrt{2}}{6} = \frac{2\sqrt{2}}{3}$

10. $\int_c (x^2+y^2)\,ds$



$\int_0^{\pi/2} (4)(2\,dt) =$ $\quad r(t) = 2\cos t\,\hat{\imath} + 2\sin t\,\hat{\jmath}$

$t \in [0, \pi/2]$

$8\int_0^{\pi/2} dt = 8\cdot\frac{\pi}{2} = 4\pi$ $\quad r'(t) = -2\sin t\,\hat{\imath} + 2\cos t\,\hat{\jmath}$

$\|r'(t)\| = \sqrt{4\sin^2 t + 4\cos^2 t} = \sqrt{4} = 2$

11. $\int_c \vec{F}\cdot d\vec{r}$ $\quad ds = \|r'(t)\|dt = 2dt$

$x = 2\cos t$

$y = 2\sin t$

$dr = -4\sin t\,\hat{\imath} + \cos t\,\hat{\jmath} + 2t\,\hat{k}$

$x = 4\cos t \quad y = \sin t \quad z = t^2$

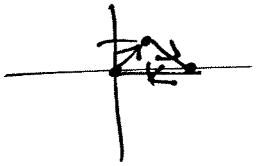
$x^2 \;\; 16\cos^2 t \quad y^2 = \sin^2 t \quad z^2 = t^4$

$\int_0^{\pi/2} -64\cos^2 t\sin t + \sin^2 t\cos t + 2t\cdot t^4\,dt$

$+\frac{64}{3}\cos^3 t + \frac{1}{3}\sin^3 t + \frac{1}{3}t^6 \Big|_0^{\pi/2}$

$\frac{64}{3}(0-1) + \frac{1}{3}(1-0) + \frac{1}{3}(\pi/2)^6 = -\frac{64}{3} + \frac{1}{3} + \frac{\pi^6}{192} = -21 + \frac{\pi^6}{192}$

12. $\int_c y\,dx - x\,dy$



$r_1(t) = t\hat{\imath} + t\hat{\jmath} \quad t \in [0,1]$

$r_2(t) = (+t+1)\hat{\imath} + (-t+1)\hat{\jmath} \quad t \in [0,1]$

$r_3(t) = (-2t+2)\hat{\imath} + 0\hat{\jmath} \quad t \in [0,1]$

$r_1'(t) = \hat{\imath} + \hat{\jmath}$

$r_2'(t) = 1\hat{\imath} - 1\hat{\jmath}$

$r_3'(t) = -2\hat{\imath}$

$\int_0^1 t(1)\,dt - t(1)\,dt = 0$

$\int_0^1 (1-t)(1)dt - (t+1)(-1)dt = \int_0^1 1-t+t+1\,dt = \int_0^1 2\,dt = 2$

$\int_0^1 (0)\,dt - (-2t+2)(0)\,dt = 0$

$\int_c y\,dx - x\,dy = 2$

13. $\int_c [2(x+y)\hat{\imath} + 2(x+y)\hat{\jmath}]\cdot dr$

$r(t) = 3t\hat{\imath} + 8t\hat{\jmath} \quad [0,1] \ni t$

$r'(t) = 3\hat{\imath} + 8\hat{\jmath}$

$\int_0^1 2(3t+8t)3\,dt + 2(3t+8t)8\,dt =$

$\int_0^1 66t + 176t\,dt = \int_0^1 242t\,dt = 121t^2\Big|_0^1 = 121$

14. $\int_c F\cdot dr \quad \vec{F}(x,y) = xy^2\hat{\imath} + 2x^2y\hat{\jmath}$

$r(t) = t\hat{\imath} + \frac{1}{t}\hat{\jmath} \quad t \in [1,3]$

$r'(t) = \hat{\imath} - \frac{1}{t^2}\hat{\jmath}$

$\int_1^3 \cancel{t}\cdot\frac{1}{t^{\cancel{2}}}(1)dt + 2\cancel{t^2}\left(\frac{1}{t}\right)\left(\frac{-1}{\cancel{t^2}}\right)dt$

$\int_1^3 \frac{1}{t} - \frac{2}{t}\,dt = \int_1^3 -\frac{1}{t}\,dt = -\ln t\Big|_1^3 = -\ln 3 + \ln 1 = -\ln 3$