Name KEY

Math 254, Exam #1, Winter 2012

Instructions: Show all work. Use exact answers unless directed otherwise (with the exception of some application problems). Problems with answers only will rarely receive full credit. Be sure to read each problem carefully and complete all parts.

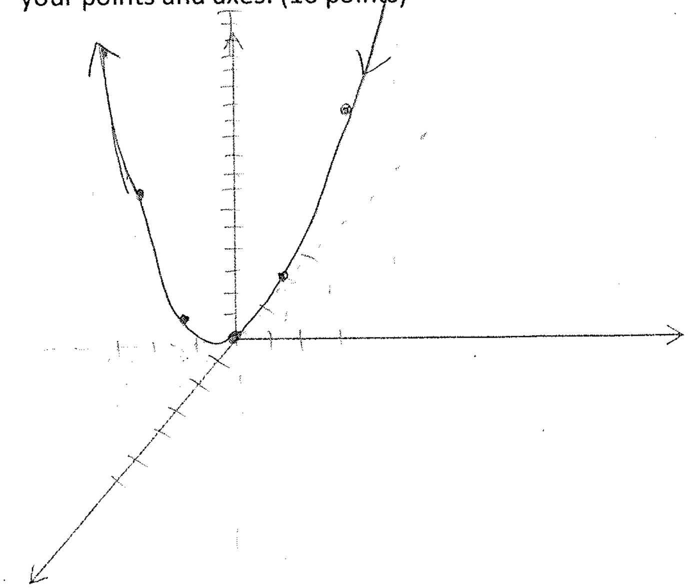
1. Find the vector-valued function that represents the intersection of the surfaces $z = x^2 + y^2$ and $x + y = 0$, using the parameter $x = t$. (5 points)

$-x = +y \qquad y = -t$

$z = (t)^2 + (-t)^2 = 2t^2$

$\boxed{r(t) = t\,\hat{i} - t\,\hat{j} + 2t^2\,\hat{k}}$

On the graph below, plot the curve using $t = \{-3, -2, -1, 0, 1, 2, 3\}$. Be sure to properly label your points and axes. (10 points)



t	x	y	z
-3	-3	3	18
-2	-2	2	8
-1	-1	1	2
0	0	0	0
1	1	-1	2
2	2	-2	8
3	3	-3	18

2. Find the following for the vector-valued function $\vec{r}(t) = \frac{1}{t}\hat{i} + 2\sin t\,\hat{j} + 2\cos t\,\hat{k}$.

   a. $\vec{r''}(t)$ (7 points)

$r'(t) = -\frac{1}{t^2}\hat{i} + 2\cos t\,\hat{j} - 2\sin t\,\hat{k}$

$\boxed{r''(t) = +\frac{2}{t^3}\hat{i} - 2\sin t\,\hat{j} - 2\cos t\,\hat{k}}$

b. $\int_1^e \vec{r}(t)dt$ (7 points)

$$\int_1^e \frac{1}{t}dt\,\hat{i} + \int_1^e 2\sin\,\hat{j}\,dt + \int_1^e 2\cos t\,\hat{k}\,dt$$

$$\ln t\Big|_1^e \hat{i} \quad -2\cos t\Big|_1^e \hat{j} \qquad 2(\sin t)\Big|_1^e \hat{k}$$

$$\boxed{1\hat{i} - 2(\cos(e) - \cos(1))\hat{j} + 2(\sin(e) - \sin(1))\hat{k}}$$

c. $D_t\left[\left\|\vec{r}(t)\right\|\right]$ (7 points)

$$\|\vec{r}(t)\| = \sqrt{\frac{1}{t^2} + 4\sin^2 t + 4\cos^2 t} = \sqrt{\frac{1}{t^2} + 4} = \sqrt{\frac{1+4t^2}{t^2}} =$$

$$\frac{\sqrt{1+4t^2}}{t}$$

$$(1+4t^2)^{1/2} \quad (1+4t^2)^{1/2}$$

$$D_t\left[\frac{(1+4t^2)^{1/2}}{t}\right] = \frac{\frac{1}{\cancel{2}}(1+4t^2)^{-1/2}\,\cancel{8}^{4}t\cdot t - (1+4t^2)^{1/2}}{t^2} = \frac{4t^2 - 1 - 4t^2}{t^2(1+4t^2)^{1/2}} =$$

$$\boxed{\frac{-1}{t^2\sqrt{1+4t^2}}}$$

3. Given the acceleration function $\vec{a}(t) = -\cos t\,\vec{j} + \sin t\,\vec{k}$, and initial conditions $\vec{v}(0) = \vec{j} + \vec{k}, \vec{r}(0) = \vec{i}$, find the velocity and position functions. (7 points)

$$\int 0\,dt\,\hat{i} + \int -\cos t\,\hat{j} + \int \sin t\,\hat{k}$$

$$c_1\hat{i} + (-\sin t + C_2)\hat{j} + (-\cos t + C_3)\hat{k} = v(t)$$

$$C_1 = 0 \qquad C_2 = 1 \qquad -1 + C_3 = 1$$

$$C_3 = 2$$

$$v(t) = (-\sin t + 1)\hat{j} + (-\cos t + 2)\hat{k}$$

$$\int 0\,dt\,\hat{i} \quad \int -\sin t + 1\,dt \quad \int -\cos t + 2\,dt$$

$$c_1\hat{i}. \quad (\cos t + t + C_2)\hat{j} + (-\sin t + 2t + C_3)\hat{k}$$

$$C_1 = 1 \qquad 1 + C_2 = 0 \qquad C_3 = 0$$

$$C_2 = -1$$

$$\boxed{r(t) = 1\hat{i} + (\cos t + t - 1)\hat{j} + (-\sin t + 2t)\hat{k}}$$

4. Find the unit tangent and unit normal vectors for the vector-valued function $\vec{r}(t) = 2\sin 4t\vec{i} + 2\cos 4t\vec{j} + 4t\vec{k}$. (10 points)

$r'(t) = 8\cos 4t\hat{i} - 8\sin 4t\hat{j} + 4\hat{k}$

$\|r'(t)\| = \sqrt{64\cos^2 4t + 64\sin^2 4t + 16} = \sqrt{64+16} = \sqrt{80} = 4\sqrt{5}$

$\vec{T}(t) = \frac{\overset{2}{\cancel{8}}\cos 4t}{\cancel{4}\sqrt{5}}\hat{i} - \frac{\overset{2}{\cancel{8}}\sin 4t}{\cancel{4}\sqrt{5}}\hat{j} + \frac{\cancel{4}\hat{k}}{\cancel{4}\sqrt{5}} =$

$\boxed{\frac{2\cos 4t}{\sqrt{5}}\hat{i} - \frac{2}{\sqrt{5}}\sin 4t\hat{j} + \frac{1}{\sqrt{5}}\hat{k}}$ unit tangent

$T'(t) = -8\sin 4t\hat{i} - 8\cos 4t\hat{j} + 0\hat{k}$

$\|T'(t)\| = \sqrt{64\sin^2 4t + 64\cos^2 4t} = \sqrt{64} = 8$

$\vec{N}(t) = \frac{-\cancel{8}\sin 4t}{\cancel{8}}\hat{i} - \frac{8\cos 4t}{8}\hat{j} = \boxed{-\sin 4t\hat{i} - \cos 4t\hat{j}}$

unit normal

5. Using the vector-valued function $\vec{r}(t) = 2\sin 4t\vec{i} + 2\cos 4t\vec{j} + 4t\vec{k}$, calculate the length of the arc on the interval [0,2π]. (7 points)

$r'(t) = 8\cos 4t\hat{i} - 8\sin 4t\hat{j} + 4\hat{k}$ from #4

$\|r'(t)\| = \sqrt{80} = 4\sqrt{5}$ ||

$\int_0^{2\pi} 4\sqrt{5}\,dt = 4\sqrt{5}t\Big|_0^{2\pi} = \boxed{8\pi\sqrt{5}}$

6. Find the curvature of the vector-valued function $\vec{r}(t) = 2\sin 4t\vec{i} + 2\cos 4t\vec{j} + 4t\vec{k}$. (7 points)

$$r'(t) = 8\cos 4t\,\hat{i} - 8\sin 4t\,\hat{j} + 4\hat{k}$$

$$r''(t) = -32\sin 4t\,\hat{i} - 32\cos 4t\,\hat{j} + 0k$$

$$r'(t) \times r''(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 8\cos 4t & -8\sin 4t & 4 \\ -32\sin 4t & -32\cos 4t & 0 \end{vmatrix} = (0 + 128\cos 4t)\hat{i} - (0 + 128\sin^2 4t)\hat{j} + (-256\cos^2 4t - 256\sin^2 4t)\hat{k}$$

$$= 128\cos 4t\,\hat{i} - 128\sin 4t\,\hat{j} - 256\hat{k} = 128(\cos 4t\,\hat{i} - \sin 4t\,\hat{j} - 2k)$$

$$\|r'(t) \times r''(t)\| = \sqrt{\cos^2 4t + \sin^2 4t + 4} \cdot 128 = 128\sqrt{1+4} = 128\sqrt{5}$$

$$\|r'(t)\| = 4\sqrt{5}$$

$$K = \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^3} = \frac{\cancel{128}^{\cancel{32}\,2}\cancel{\sqrt{5}}}{\cancel{4}\sqrt{5}\cdot\cancel{4}\sqrt{5}\cdot\cancel{4}\cancel{\sqrt{5}}} = \boxed{\frac{2}{5}}$$

7. Sketch the level curves for the function $f(x,y) = \sqrt{9 - x^2 - y^2}$, using the values $c = 0,1,2,3$ for z. Given the information from the level curves, what does this shape look like in 3D? Describe or sketch. (10 points)

$$z^2 = 9 - x^2 - y^2$$

$$x^2 + y^2 = 9 - z^2 \qquad \text{circle}$$

$z = 0$

$x^2 + y^2 = 9 - 0 = 9$

$z = 1$

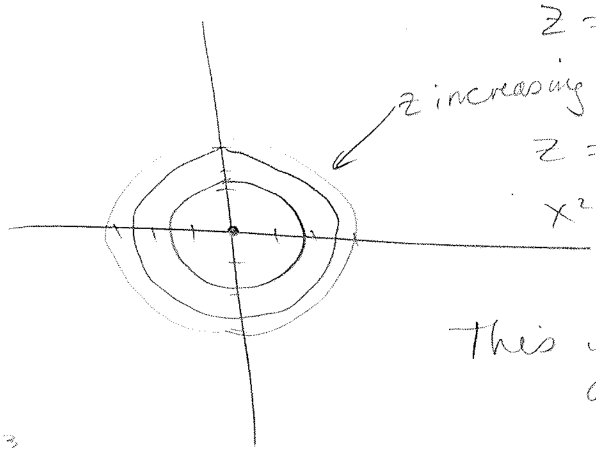
$x^2 + y^2 = 9 - 1 = 8$

$z = 2$

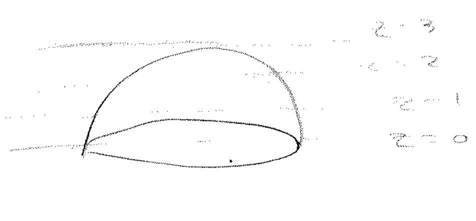
z increasing $\quad x^2 + y^2 = 9 - 4 = 5$

$z = 3$

$x^2 + y^2 = 9 - 9 = 0$



This is the top half of a sphere of radius 3



8. Find the limits or prove that the limit does not exist. (5 points each)

a. $\lim_{(x,y)\to(1,-1)} \frac{x^2y}{1+xy^2}$ $\quad \frac{1^2(-1)}{1+(1)(-1)^2} = \frac{-1}{1+1} = \boxed{\frac{-1}{2}}$

b. $\lim_{(x,y)\to(0,0)} \frac{x-y}{\sqrt{x}-\sqrt{y}} \cdot \frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}+\sqrt{y}} = \frac{\cancel{(x-y)}(\sqrt{x}+\sqrt{y})}{\cancel{x-y}}$

$\lim_{(x,y)\to(0,0)} \sqrt{x}+\sqrt{y} = \boxed{0}$

c. $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{\sqrt{x^2+y^2}}$

let $x=0$

$\lim_{y\to 0} \frac{-y^2}{\sqrt{y^2}} = \frac{-y^2}{y} = -y = 0$

let $y = kx$

$\lim_{x\to 0} \frac{x^2-k^2x^2}{\sqrt{x^2+k^2x^2}} = \frac{x^2(1-k^2)}{\sqrt{x^2(1+k^2)}} = \frac{x^{\cancel{2}}(1-k^2)}{\cancel{x}\sqrt{1+k^2}} =$

$\lim_{x\to 0} \frac{x(1-k^2)}{\sqrt{1+k^2}} = \boxed{0}$

or try polar

$\lim_{(r,\theta)\to(0,0)} \frac{r^2\cos^2\theta - r^2\sin^2\theta}{r} = \frac{r^{\cancel{2}}(\cos 2\theta)}{\cancel{r}} = \lim_{r\to 0} r\cos 2\theta = \boxed{0}$

d. $\lim_{(x,y)\to(0,0)} \frac{x^3 y}{x^6+y^2}$

let $x=0$

$\lim_{y\to 0} \frac{0\cdot y}{0+y^2} = \frac{0}{y^2} = 0$

let $y=kx^3$

$\lim_{x\to 0} \frac{x^3\cdot kx^3}{x^6+k^3x^6} = \frac{k\cancel{x^6}}{\cancel{x^6}(1+k^3)} = \frac{k}{1+k^3} \neq 0$ unless $k=0$

$\boxed{DNE}$

e. $\lim_{(x,y,z)\to(0,0,0)} \arctan\left[\frac{1}{x^2+y^2+z^2}\right]$

switch to spherical

$\lim_{(\rho,\varphi,\theta)\to(0,\varphi,\theta)} \arctan\left[\frac{1}{\rho^2}\right] = \arctan(+\infty) = \boxed{\pi/2}$

9. Find all first partial derivatives of $z=e^y \sin xy$. Then find z_{xyx}. (7 points)

$z_x = e^y\cos(xy)\cdot y$

$z_y = e^y\sin(xy) + e^y\cos(xy)\cdot x = e^y[\sin(xy) + x\cos(xy)]$

$z_{xy} = e^y\cos(xy)\cdot y + -e^y\sin(xy)\cdot xy + e^y\cos(xy)\cdot 1 =$

$e^y[y\cos(xy) - xy\sin(xy) + \cos(xy)]$

$z_{xyx} = e^y[-y\sin(xy)\cdot y - y\sin(xy) - xy\cos(xy)\cdot y - \sin(xy)\cdot y] =$

$\boxed{= -e^y\cdot y[y\sin(xy) + 2\sin(xy) + xy\cos(xy)]}$

10. Using the formula for the gradient in spherical coordinates $\nabla = \left\langle \frac{\partial}{\partial\rho}, \frac{1}{\rho}\cdot\frac{\partial}{\partial\varphi}, \frac{1}{\rho\sin\varphi}\cdot\frac{\partial}{\partial\theta} \right\rangle$, calculate the gradient of the function $f(\rho,\varphi,\theta) = \rho^2\sin^2\varphi + 2\rho\tan\theta$. (10 points)

$\frac{\partial f}{\partial \rho} = 2\rho\sin^2\varphi + 2\tan\theta$

$\frac{\partial f}{\partial \varphi} = 2\rho^2 \sin\varphi\cos\varphi \qquad \frac{1}{\rho}\frac{\partial f}{\partial \varphi} = 2\rho\sin\varphi\cos\varphi = 2\rho\sin 2\varphi$ (optional last step)

$\frac{\partial f}{\partial \theta} = 2\rho\sec^2\theta \qquad \frac{1}{\rho\sin\varphi}\frac{\partial f}{\partial \theta} = \frac{1}{\cancel{\rho}\sin\varphi}2\cancel{\rho}\sec^2\theta = 2\csc\varphi\sec^2\theta$

$$\boxed{\nabla f(\rho,\varphi,\theta) = \langle 2\rho\sin^2\varphi + 2\tan\theta,\ 2\rho\sin 2\varphi,\ 2\csc\varphi\sec^2\theta \rangle}$$

11. Find the total differential for $w = 2z^3\sin xy$ then use the value of w(π,1/2,1) to approximate the value at w(7π/6,0.4,1.05). (7 points)

$dx = \pi/6$

$dy = -.1$

$dz = .05$

$dw = w_z\,dz + w_x\,dx + w_y\,dy$

$w_z = 6z^2\sin xy$

$w_x = 2z^3\cos(xy)\cdot y$

$w_y = 2z^3\cos(xy)\cdot x$

$dw = 6z^2\sin xy\cdot(.05) + 2z^3\cos(xy)\cdot y\,(\pi/6) + 2z^3\cos(xy)\cdot x\,(-.1)$

$6(1)^2\underbrace{\sin(\pi/2)}_{(1)}(.05) + \underbrace{2(1)^3}_{2}\underbrace{\cos(\pi/2)}_{(0)}\cdot\underbrace{1/2(\pi/6)}_{\pi/12} + \underbrace{2(1)^3\cos(\pi/2)(\pi)(-.1)}_{2\,(0)\,(\pi)\,(-.1)\,=0}$

$= 0$

$6 \times .05 = .3$

$\boxed{dw = .3}$

12. Find the partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for $x\ln y + y^2 z + z^2 = 8$. Do it the long way and not the short-cut formulas. (8 points)

$$\ln y + y^2 z_x + 2z \cdot z_x = 0$$

$$z_x(y^2 + 2z) = -\ln y \Longrightarrow \boxed{z_x = \frac{-\ln y}{y^2+2z}}$$

$$x \cdot \frac{1}{y} + 2yz + y^2 z_y + 2z \cdot z_y = 0$$

$$z_y(y^2+2z) = -\frac{x}{y} - 2yz \Rightarrow \boxed{z_y = \frac{-\frac{x}{y} - 2yz}{y^2+2z}}$$

clear frac. $\frac{y}{y} \Rightarrow z_y = \frac{-x-2y^2z}{y^3+2zy}$

13. Use the chain rule to find $\frac{dw}{dt}$ for $w = \frac{x^2}{y}, x = t^2, y = t+1$. Be sure your final answer contains only one variable. (8 points)

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} = \frac{2t^2}{t+1} \cdot 2t + \frac{-t^4}{(t+1)^2} \cdot 1 =$$

$$\frac{\partial w}{\partial x} = \frac{2x}{y} = \frac{2t^2}{t+1} \qquad \frac{dx}{dt} = 2t$$

$$\frac{\partial w}{\partial y} = \frac{-x^2}{y^2} = \frac{-t^4}{(t+1)^2} \qquad \frac{dy}{dt} = 1$$

$$\boxed{\frac{4t^3}{t+1} - \frac{t^4}{(t+1)^2}}$$

Or $\frac{4t^3(t+1) - t^4}{(t+1)^2} =$

$$\frac{4t^4 + 4t^3 - t^4}{(t+1)^2} = \frac{3t^4+4t^3}{(t+1)^2}$$

14. Find the directional derivative of the function at the indicated point, in the indicated direction $f(x,y,z) = xy + yz + xz; P(1,1,1); \vec{v} = 2\vec{i} + \vec{j} - \vec{k}$. (8 points)

$$\|\vec{v}\| = \sqrt{4+1+1} = \sqrt{6}$$

$$\vec{u} = \frac{2}{\sqrt{6}}\hat{i} + \frac{1}{\sqrt{6}}\hat{j} - \frac{1}{\sqrt{6}}\hat{k}$$

$$\nabla f = (y+z)\hat{i} + (x+z)\hat{j} + (x+y)\hat{k}$$

$$= 2\hat{i} + 2\hat{j} + 2\hat{k}$$

$$\langle 2, 2, 2 \rangle \cdot \left\langle \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}} \right\rangle =$$

$$\frac{4}{\sqrt{6}} + \cancel{\frac{2}{\sqrt{6}}} - \cancel{\frac{2}{\sqrt{6}}} = \boxed{\frac{4}{\sqrt{6}}}$$