MTH 265, Quiz #15, Summer 2024 Name KEY

Instructions: Show all work. Use exact answers unless otherwise asked to round.

1. Evaluate $\int_C \vec{F}\cdot d\vec{r}$ for $\vec{F}(x,y,z) = -y\hat{\imath} + x\hat{\jmath} - 2\hat{k}$ for the boundary of the surface $S: z^2 = x^2 + y^2, 0 \le z \le 4$, oriented downward, using Stokes' Theorem.

$z = \sqrt{x^2+y^2}$ $\quad G = \sqrt{x^2+y^2} - z$ $\quad \nabla G = \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, -1 \right\rangle$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y & x & -z \end{vmatrix} = (0-0)\hat{\imath} - (0-0)\hat{\jmath} + (1+1)\hat{k} = 2\hat{k}$$

$$(\vec{\nabla}\times\vec{F})\cdot\nabla G = -2$$

$$\int_0^{2\pi}\int_0^4 -2r\,dr\,d\theta = \int_0^{2\pi} -r^2\Big|_0^4 d\theta = -16\cdot 2\pi = -32\pi$$

2. Evaluate the flux $\iint_S \vec{F}\cdot d\vec{S}$ for $\vec{F}(x,y,z) = (\cos z + xy^2)\hat{\imath} + xe^{-z}\hat{\jmath} + (\sin y + x^2 z)\hat{k}$, where S is the surface of the solid bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 4$.

$\vec{\nabla}\cdot\vec{F} = y^2 + 0 + x^2 = r^2$

$z = r^2$

$4 = r^2 \Rightarrow r = 2$

$$\int_0^{2\pi}\int_0^2\int_{r^2}^4 r^2\cdot r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^2\int_{r^2}^4 r^3\,dz\,dr\,d\theta =$$

$$\int_0^{2\pi}\int_0^2 r^3 z\Big|_{r^2}^4 dr\,d\theta = \int_0^{2\pi}\int_0^2 r^3(4-r^2)\,dr\,d\theta = \int_0^{2\pi}\int_0^2 4r^3 - r^5\,dr\,d\theta =$$

$$\int_0^{2\pi} r^4 - \frac{1}{6}r^6\Big|_0^2 d\theta = \int_0^{2\pi} 16 - \frac{32}{3}\,d\theta = 2\pi\left(\frac{16}{3}\right) = \frac{32\pi}{3}$$

3. Evaluate $\int_C \vec{F}\cdot d\vec{r}$ for $\vec{F}(x,y,z) = -2yz\hat{\imath} + y\hat{\jmath} + 3x\hat{k}$ for the boundary of the surface $S: z = 5 - x^2 - y^2, z \geq 1$, oriented upward, using Stokes' Theorem.

$G = z - 5 + x^2 + y^2 \qquad \vec{\nabla} G = \langle 2x, 2y, 1\rangle \qquad 1 = 5 - x^2 - y^2$

$4 = x^2 + y^2 \Rightarrow r = 2$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -2yz & y & 3x \end{vmatrix} = (0-0)\hat{\imath} - (3-2y)\hat{\jmath} + (0+2z)\hat{k}$$

$$(\vec{\nabla}\times\vec{F})\cdot\nabla G = 2y(2y-3) + 2z = 4y^2 - 6y + 2(5 - x^2 - y^2) =$$

$$4y^2 - 6y + 10 - 2x^2 - 2y^2 = 2y^2 - 6y + 10 - 2x^2$$

$$\int_0^{2\pi}\int_0^2 [2r^2\sin^2\theta - 6r\sin\theta + 10 - 2r^2\cos^2\theta]\, r\,dr\,d\theta =$$

$$\int_0^{2\pi}\int_0^2 2r^3\cos 2\theta - 6r^2\sin\theta + 10r\,dr\,d\theta = \int_0^{2\pi} \tfrac{1}{2}r^4\cos 2\theta - 2r^3\sin\theta + 5r^2\Big|_0^2 d\theta =$$

$$\int_0^{2\pi} 8\cos 2\theta - 16\sin\theta + 10\,d\theta = 4\sin 2\theta + 16\cos\theta + 20\theta\Big|_0^{2\pi} = 40\pi$$

4. Evaluate the flux $\iint_S \vec{F}\cdot d\vec{S}$ for $\vec{F}(x,y,z) = x^2\hat{\imath} + xy\hat{\jmath} + z\hat{k}$, where S is the surface of the solid bounded by the paraboloid $z = 4 - x^2 - y^2$ and the xy-plane using the Divergence Theorem.

$4 - r^2$

$$\vec{\nabla}\cdot\vec{F} = 2x + x + 1 = 3x + 1$$

$$\int_0^{2\pi}\int_0^2\int_0^{4-r^2} (3r\cos\theta + 1)\, r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^2\int_0^{4-r^2} 3r^2\cos\theta + r\,dz\,dr\,d\theta =$$

$$\int_0^{2\pi}\int_0^2 (3r^2\cos\theta + r)z\Big|_0^{4-r^2} dr\,d\theta = \int_0^{2\pi}\int_0^2 12r^2\cos\theta + 4r - 3r^4\cos\theta - r^3\,dr\,d\theta$$

$$= \int_0^{2\pi} 4r^3\cos\theta + 2r - \tfrac{3}{5}r^5\cos\theta - \tfrac{1}{4}r^4\Big|_0^2 d\theta = \int_0^{2\pi} \tfrac{64}{5}\cos\theta + 4\,d\theta =$$

$$\tfrac{64}{5}\sin\theta + 4\theta\Big|_0^{2\pi} = 8\pi$$