6/4/2024

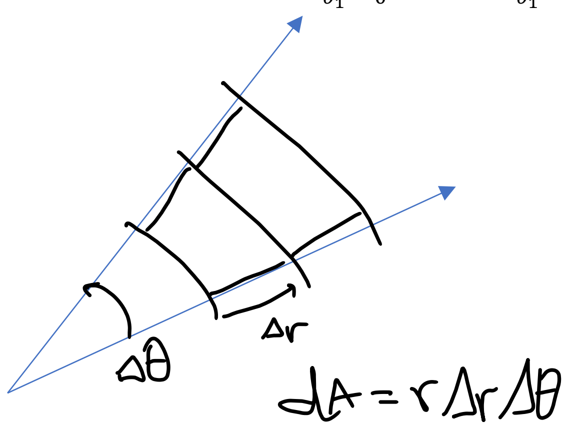
Integrals in Polar Coordinates

Area formula in one-variable for polar coordinates: where does this come from?

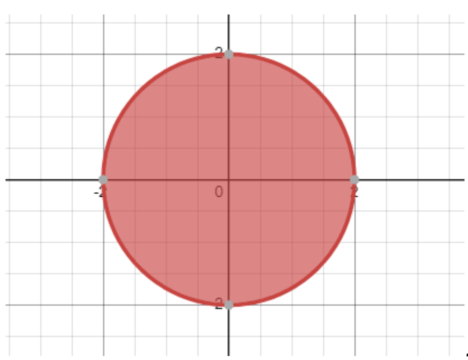
$$A = \frac{1}{2}\int_{\theta_1}^{\theta_2} [r(\theta)]^2 d\theta$$

In a double integral:
The area of a region in polar coordinates is given by

$$A = \int_{\theta_1}^{\theta_2}\int_{0}^{r(\theta)} dA = \int_{\theta_1}^{\theta_2}\int_{0}^{r(\theta)} r dr d\theta = \int_{\theta_1}^{\theta_2} \left[\frac{1}{2}r^2\right]_0^{r(\theta)} d\theta$$



The regions we tend to work with in polar (or switch to polar) will be regions that are cleaner in polar form instead of rectangular form.

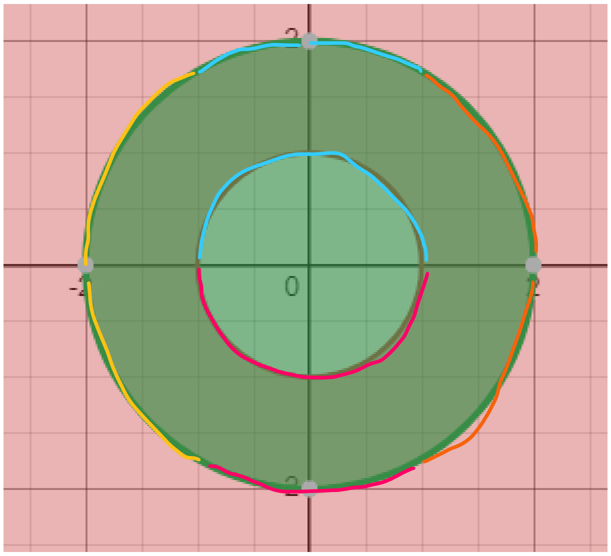


$x^2 + y^2 \leq 4$

Your upper limit in rectangular would be $y = \sqrt{4 - x^2}$ and the lower limit would be $y = -\sqrt{4 - x^2}$. You will (probably) need trig substitution to finish the integration.

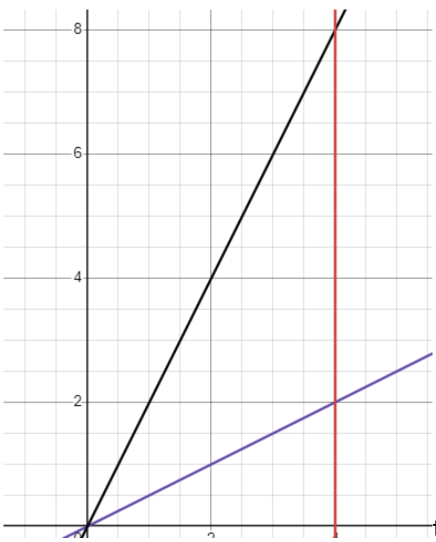
On the other hand, in polar form, this circle is $r \leq 2$. So the limits are 0 and 2, which are constant.

Another example is between two circles.

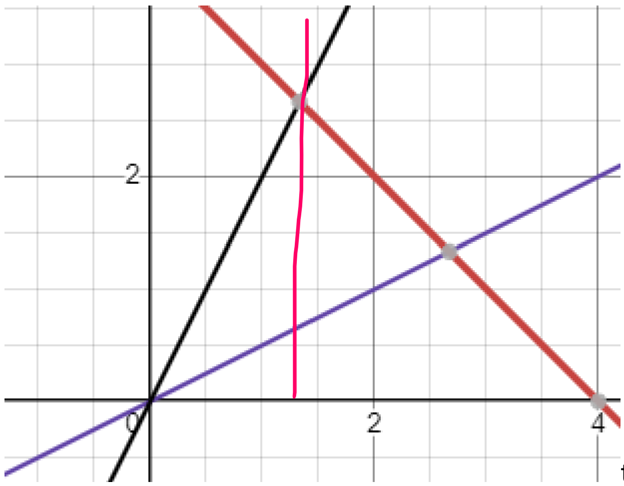


outside the circle of radius 1, and inside the circle of radius 2. If you try to setup this region in rectangular form, you need 4 integrals, maybe you can reduce to 2 if you can apply symmetry. And, the bounds are like the previous example, and you'll need trig sub to do the step with x.

In polar the limits are just between two circles of constant radius, so the bounds would be between 1 and 2 in r.



the lines going through the origin are constants in theta.

Consider: 

this region needs two integrals in rectangular, but in polar, I have just one integral and one outside polar function.

When you convert an integral in x and y (already set up) into polar, the function being integrated can be converted to polar form algebraically, but the limits of integration will need to be converted geometrically: you should graph the region, and then re-setup the integral limits fresh. You can't just convert the limits algebraically. (that will not work).

Example.
Evaluate $\int\int_D x^2 y dA$ where D is the top half of the disk with the center at the origin and a radius of 5.

$$x^2 + y^2 \le 25$$

$$\int_0^{2\pi} \int_0^5 r^2 \cos^2\theta \, r\sin\theta \, r dr d\theta = \int_0^{2\pi} \int_0^5 r^4 \cos^2\theta \sin\theta \, dr d\theta = \int_0^{2\pi} \frac{r^5}{5}\bigg|_0^5 \cos^2\theta \sin\theta \, d\theta =$$

$$\int_0^{2\pi} 625\cos^2\theta \sin\theta \, d\theta = -\frac{625}{3}\cos^3\theta\bigg|_0^{2\pi} = 0$$

Example.
Evaluate the integral $\int\int_D \frac{y^2}{x^2+y^2} dA$ where D is the region between the circles $x^2 + y^2 = 1, x^2 + y^2 = 9$.

$$\int_0^{2\pi} \int_1^3 \frac{r^2 \sin^2\theta}{r^2} r dr d\theta = \int_0^{2\pi} \int_1^3 r\sin^2\theta \, dr d\theta = \int_0^{2\pi} \frac{1}{2}r^2\bigg|_1^3 \sin^2\theta \, d\theta = 4\int_0^{2\pi} \sin^2\theta \, d\theta =$$

$$2\int_0^{2\pi} 1 - \cos 2\theta \, d\theta = 2\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{2\pi} = 4\pi$$

Example.
Find the volume below the paraboloid $z = 18 - 2x^2 - 2y^2$ and above the xy-plane.

$$0 = 18 - 2x^2 - 2y^2$$
$$2x^2 + 2y^2 = 18$$
$$x^2 + y^2 = 9$$

$$z = 18 - 2(x^2 + y^2) = 18 - 2r^2$$

$$\int_0^{2\pi} \int_0^3 (18 - 2r^2) r dr d\theta = \int_0^{2\pi} \int_0^3 18r - 2r^3 dr\, d\theta = \int_0^{2\pi} \left[9r^2 - \frac{1}{2} r^4\right]_0^3 d\theta =$$

$$\int_0^{2\pi} 81 - \frac{81}{2} d\theta = \int_0^{2\pi} \frac{81}{2} d\theta = \frac{81}{2} \theta \Big|_0^{2\pi} = 81\pi$$

Example.

Find the volume above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = 1$

$$z^2 = x^2 + y^2$$
$$2x^2 + 2y^2 = 1$$
$$x^2 + y^2 = \frac{1}{2}$$
$$r = \frac{1}{\sqrt{2}}$$
$$z = \sqrt{1 - x^2 - y^2} = \sqrt{1 - r^2}$$
$$z = r$$

$$\int_0^{2\pi} \int_0^{\frac{1}{\sqrt{2}}} (sphere - cone) r dr d\theta = \int_0^{2\pi} \int_0^{\frac{1}{\sqrt{2}}} \left(\sqrt{1 - r^2} - r\right) r dr d\theta = \int_0^{2\pi} \int_0^{\frac{1}{\sqrt{2}}} r\sqrt{1 - r^2} - r^2 dr d\theta$$

$$= \int_0^{2\pi} -\frac{1}{2}\left(\frac{2}{3}\right)(1 - r^2)^{\frac{3}{2}} - \frac{1}{3} r^3 \Big|_0^{\frac{1}{\sqrt{2}}} d\theta = -\frac{1}{3}\left(\frac{1}{2}\right)^{\frac{3}{2}} - \frac{1}{6\sqrt{2}} + \frac{1}{3}(1) \int_0^{2\pi} d\theta = \left(\frac{1}{3} - \frac{1}{3\sqrt{2}}\right) 2\pi$$

Example.

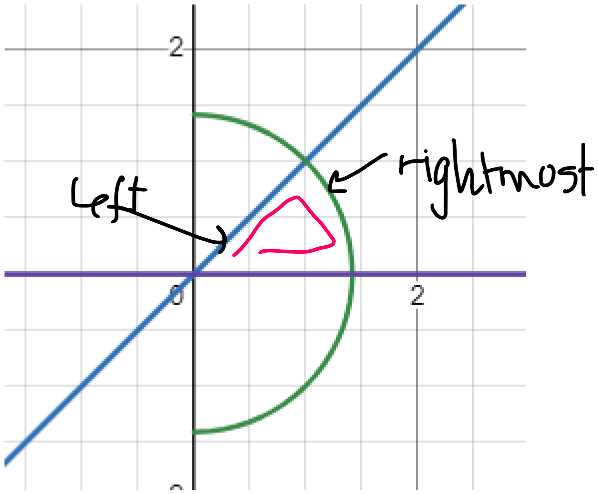
Convert the integral in rectangular to polar form.

$$\int_0^1 \int_y^{\sqrt{2 - y^2}} (x + y) dx dy$$

$$\int_{\square}^{\square} \int_{\square}^{\square} (r \cos\theta + r \sin\theta) r dr d\theta$$

$$x = y, x = \sqrt{2 - y^2} \to x^2 = 2 - y^2 \to x^2 + y^2 = 2$$
$$y = 0, y = 1$$

$$y = \sqrt{2 - y^2} \to y^2 = 2 - y^2 \to 2y^2 = 2 \to y^2 = 1 \to y = 1$$



$$r = \sqrt{2}$$
$$x = y$$
$$1 = \frac{y}{x}$$
$$\arctan\left(\frac{y}{x}\right) = \theta \to \arctan(1) = \theta = \frac{\pi}{4}$$

$$\int_0^{\frac{\pi}{4}}\int_0^{\sqrt{2}}(r\cos\theta + r\sin\theta)rdrd\theta = \int_0^{\frac{\pi}{4}}\int_0^{\sqrt{2}} r^2(\cos\theta + \sin\theta)drd\theta = \int_0^{\frac{\pi}{4}}\frac{1}{3}r^3\Big|_0^{\sqrt{2}}(\cos\theta + \sin\theta)d\theta =$$

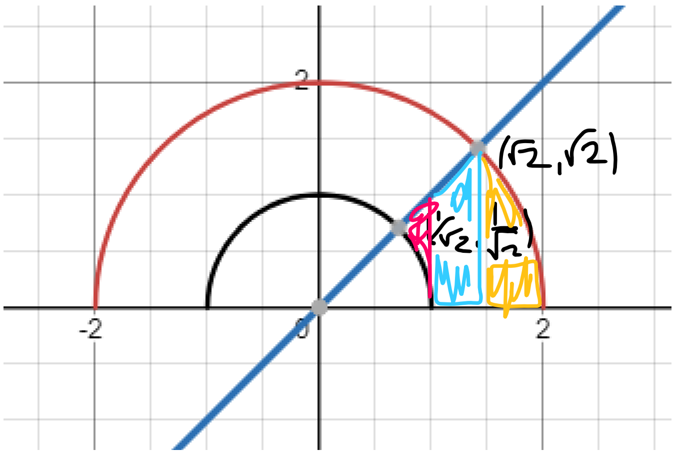
$$\int_0^{\frac{\pi}{4}}\frac{2\sqrt{2}}{3}(\cos\theta + \sin\theta)d\theta = \frac{2\sqrt{2}}{3}[\sin\theta - \cos\theta]_0^{\frac{\pi}{4}} = \frac{2\sqrt{2}}{3}\left[\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} - 0 + 1\right] = \frac{2\sqrt{2}}{3}$$

Example.
Use polar coordinates to combine

$$\int_{\frac{1}{\sqrt{2}}}^{1}\int_{\sqrt{1-x^2}}^{x} xydydx + \int_1^{\sqrt{2}}\int_0^x xydydx + \int_{\sqrt{2}}^{2}\int_0^{\sqrt{4-x^2}} xy\,dydx$$

$$\int_{\square}^{\square}\int_{\square}^{\square} xydA = \int_{\square}^{\square}\int_{\square}^{\square}(r\,cos\theta)(r\sin\theta)rdrd\theta$$

$$y = \sqrt{1-x^2}, y = \sqrt{4-x^2}$$
$$x^2 + y^2 = 1, x^2 + y^2 = 4$$
$$y = x$$



$$\int_0^{\frac{\pi}{4}}\int_1^2 (r\,cos\theta)(r\sin\theta)rdrd\theta = \int_0^{\frac{\pi}{4}}\int_1^2 r^3(cos\theta)(\sin\theta)drd\theta =$$

$$\int_0^{\frac{\pi}{4}}\frac{1}{4}r^4\Big|_1^2 \cos\theta\sin\theta\,d\theta = \int_0^{\frac{\pi}{4}}\left(4-\frac{1}{4}\right)\cos\theta\sin\theta\,d\theta = \frac{15}{4}\left(\frac{1}{2}\right)\sin^2\theta\Big|_0^{\frac{\pi}{4}} = \frac{15}{8}\left[\frac{1}{2}\right] = \frac{15}{16}$$

Example.

$$\int_{-\infty}^{\infty} e^{-x^2}dx = I$$

Trick:
Multiple the integral by another copy of itself, but in y instead of x.

$$I^2 = \int_{-\infty}^{\infty} e^{-x^2}dx\left(\int_{-\infty}^{\infty} e^{-y^2}dy\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}dydx = \int_0^{2\pi}\int_0^{\infty} e^{-r^2}rdrd\theta =$$

$$\int_0^{2\pi} -\frac{1}{2}e^{-r^2}\Big|_0^{\infty} d\theta = -\frac{1}{2}(0-1)\int_0^{2\pi} d\theta = \frac{1}{2}(2\pi) = \pi$$

$$I^2 = \pi \rightarrow I = \sqrt{\pi}$$

Next time: triple integrals.