6/20/2024

Relative and Absolute Extrema in Functions of Two Variables (14.7/4.7)
Review for Exam #3

Relative Extrema

Example. (continuing from last lecture)
Find and characterize the critical point(s) for the function $f(x, y) = xy - 2x - 2y - x^2 - y^2$

$$f_x = y - 2 - 2x$$
$$f_y = x - 2 - 2y$$

$$f_{xx} = -2$$
$$f_{yy} = -2$$
$$f_{xy} = 1$$

To find the critical point, set the first derivatives equal to zero.

$$y - 2 - 2x = 0$$
$$x - 2 - 2y = 0$$

$$y = 2 + 2x$$
$$x - 2 - 2(2 + 2x) = 0$$
$$x - 2 - 4 - 4x = 0$$
$$-3x - 6 = 0$$
$$3x = -6$$
$$x = -2$$
$$y = 2 + 2(-2) = 2 - 4 = -2$$
$$(-2, -2)$$

$$D = f_{xx}f_{yy} - \left(f_{xy}\right)^2 = (-2)(-2) - (1)^2 = 4 - 1 = 3$$

D positive means it's a minimum or a maximum, and since f_{xx} is negative, the graph is concave down, and so it's a maximum.

Example.
Find and characterize the critical point(s) for the function $f(x, y) = y^3 + 3x^2y - 6x^2 - 6y^2 + 2$

$$f_x = 6xy - 12x$$
$$f_y = 3y^2 + 3x^2 - 12y$$

$$f_{xx} = 6y - 12$$
$$f_{yy} = 6y - 12$$
$$f_{xy} = 6x$$

Find the critical points: set the first derivatives equal to 0.

$$6xy - 12x = 0$$

$$3y^2 + 3x^2 - 12y = 0$$

$$6x(y - 2) = 0$$

Either, x=0 or y=2.

Plug in x=0 into the f_y derivative.

$$3y^2 + 0 - 12y = 0$$
$$3y(y - 4) = 0$$
$$y = 0, y = 4$$

Critical points are : $(0,0), (0,4)$

Plug in y=2 into the f_y derivative:

$$3(2)^2 + 3x^2 - 12(2) = 0$$
$$12 + 3x^2 - 24 = 0$$
$$3x^2 = 12$$
$$x^2 = 4$$
$$x = \pm 2$$

Critical points are : $(2,2), (-2,2)$

Test in the D equation to characterize the points:

Test (0,0)

$$f_{xx} = 6y - 12$$
$$f_{yy} = 6y - 12$$
$$f_{xy} = 6x$$
$$D(0,0) = (-12)(-12) - 0^2 = 144$$

Maximum, since D is positive and f_{xx} is negative.

Test (0,4)

$$D(0,4) = (12)(12) - 0^2 = 144$$

Minimum, since D is positive and f_{xx} is also positive.

Test (2,2)

$$D(2,2) = (0)(0) - 12^2 = -144$$

Saddle point since D is negative.

Test (-2,2)

$$D(-2,2) = (0)(0) - (-12)^2 = -144$$

Saddle point since D is negative.

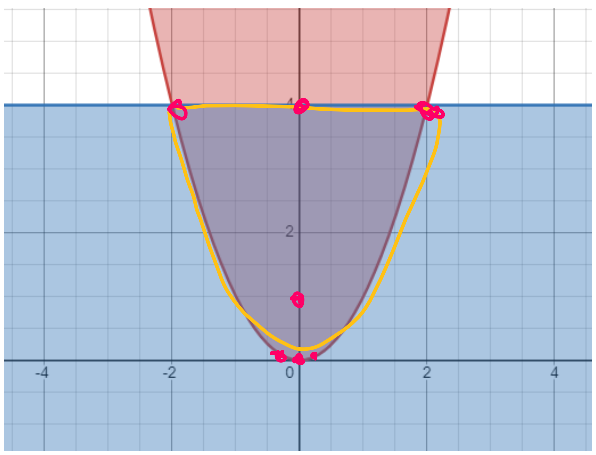
Absolute Extrema

Steps:

Find the critical point(s) on the function, and determine if they are inside the bounding region. Discard any that are not.
Test the boundary conditions (functions on the boundary of the region), plug into the surface (reduces the function to a single variable), and then look for one-variable extrema on those boundaries.
Find the intersections of the boundaries (corner points).

Test all critical points and corner points. The biggest is the max, and the smallest is the absolute min.

Example.
Find the absolute extrema of the function $f(x,y) = 3x^2 + 2y^2 - 4y$ on the region bounded by $y \geq x^2, y \leq 4$



1) Find the critical point(s) for the function.

$$f_x = 6x$$
$$f_y = 4y - 4$$

$$6x = 0, x = 0$$
$$4y - 4 = 0, 4y = 4, y = 1$$

Critical point is (0,1)
Check, is this in the region? Yes.

2) Test the boundaries.
   a) Test $y = x^2$, substitute into the surface to see how it behaves along that boundary

$$f(x, x^2) = 3x^2 + 2(x^2)^2 - 4(x^2) = 3x^2 + 2x^4 - 4x^2 = 2x^4 - x^2$$

$$f'(x, x^2) = 8x^3 - 2x = 0$$
$$2x(4x^2 - 1) = 0$$
$$x = 0, 4y^2 = 1, x^2 = \frac{1}{4}, x = \pm\frac{1}{2}$$

$$(0,0), \left(\frac{1}{2}, \frac{1}{4}\right), \left(-\frac{1}{2}, \frac{1}{4}\right)$$

   b) Test y=4

$$f(x,4) = 3x^2 + 2(4^2) - 4(4) = 3x^2 + 32 - 16 = 3x^2 + 16$$
$$f'(x,4) = 6x$$
$$6x = 0, x = 0$$
$$(0,4)$$

3) Test the corner points
   Where do the boundaries intersect:

$$y = x^2, y = 4$$
$$x^2 = 4, x = \pm 2$$

$$(2,4)(-2,4)$$

Test all the points we found in the original function to find the biggest and the smallest z-value.

(x,y)	$f(x,y = 3x^2 + 2y^2 - 4y)$
$(0,1)$	$0 + 2 - 4 =$ -2 Minumum
$(0,0)$	$0 + 0 - 0 = 0$
$\left(\frac{1}{2}, \frac{1}{4}\right)$	$\frac{3}{4} + \frac{1}{8} - 1 = -\frac{1}{8}$
$\left(-\frac{1}{2}, \frac{1}{4}\right)$	$-\frac{1}{8}$
$(0,4)$	$0 + 32 - 16 = 16$
$(-2,4)$	$12 + 32 - 16 =$ 28 Maximum
$(2,4)$	28 Maximum

Example.
Find the absolute max and min of the function $f(x,y) = 2x^3 + y^4$ on the region inside the unit circle ($x^2 + y^2 \leq 1$).

1) Critical point(s)

$$f_x = 6x^2$$
$$f_y = 4y^3$$

   Critical point is (0,0)

2) Test the boundaries

$$x^2 + y^2 = 1$$
$$y^2 = 1 - x^2$$

$$f\left(x, \sqrt{1 - x^2}\right) = 2x^3 + (1 - x^2)^2 = 2x^3 + 1 - 2x^2 + x^4$$
$$f'\left(x, \sqrt{1 - x^2}\right) = 6x^2 - 4x + 4x^3$$

$$4x^3 + 6x^2 - 4x = 0$$
$$2x(2x^2 + 3x - 2) = 0$$
$$2x(2x - 1)(x + 2) = 0$$
$$x = 0, \frac{1}{2}, -2$$

$$(0,1), (0,-1), \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$$

No corner points

Test all the points

(x, y)	$f(x, y) = 2x^3 + y^4$
$(0,0)$	$0 + 0 = 0$ Minimum
$(0,1)$	$0 + 1 = 1$ Maximum
$(0,-1)$	$0 + 1 = 1$ Maximum
$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$	$2\left(\frac{1}{2}\right)^3 + \left(\frac{\sqrt{3}}{2}\right)^4 = \frac{13}{16}$
$\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$	$\frac{13}{16}$

Lagrange Multipliers is in section (14.8/4.8), and we are not covering that in this course, and it’s not the exam.

A method of dealing with functions of two or more variables which are subject to constraints.