6/18/2024

Implicit Partial Differentiation
Chain Rule (14.5/4.5)

Chain Rule

Typically, we have a function, $f(x,y)$ or $f(x,y,z)$ and the variables can themselves be defined in terms of one or more variables, for example: $x(t), y(t), z(t), etc.$ or $x(t,s), y(t,s), z(t,s), etc.$

Start with case with the variables being themselves in terms of only one variable.

If $z = f(x,y)$, and $x = x(t), y = y(t)$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\left(\frac{dx}{dt}\right) + \frac{\partial z}{\partial y}\left(\frac{dy}{dt}\right)$$

If $w = f(x,y,z), and\ x = x(t), y = y(t), z = z(t)$:

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\left(\frac{dx}{dt}\right) + \frac{\partial w}{\partial y}\left(\frac{dy}{dt}\right) + \frac{\partial w}{\partial z}\left(\frac{dz}{dt}\right)$$

If $z = f(x,y)$, and $x = x(t,s), y = y(t,s)$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\left(\frac{\partial x}{\partial t}\right) + \frac{\partial z}{\partial y}\left(\frac{\partial y}{\partial t}\right)$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\left(\frac{\partial x}{\partial s}\right) + \frac{\partial z}{\partial y}\left(\frac{\partial y}{\partial s}\right)$$

If $w = f(x,y,z)$ and $x = x(t,s), y = y(t,s), z = z(t,s)$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\left(\frac{\partial x}{\partial t}\right) + \frac{\partial w}{\partial y}\left(\frac{\partial y}{\partial t}\right) + \frac{\partial w}{\partial z}\left(\frac{\partial z}{\partial t}\right)$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\left(\frac{\partial x}{\partial s}\right) + \frac{\partial w}{\partial y}\left(\frac{\partial y}{\partial s}\right) + \frac{\partial w}{\partial z}\left(\frac{\partial z}{\partial s}\right)$$

In general, I prefer that the final result, if you are not evaluating at a particular point, still have a function of one or two variables, NOT 4 or five variables.

Do the substitutions after taking all the derivatives. If you do it first, you'll never use the chain rule in this way.
You don't have to simplify.

Example.
Find $\frac{dz}{dt}$ using the chain rule for $z = x^2y + 3xy^4$. And $x(t) = \sin 2t\,, y(t) = \cos t$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\left(\frac{dx}{dt}\right) + \frac{\partial z}{\partial y}\left(\frac{dy}{dt}\right)$$

$\frac{\partial z}{\partial x} = 2xy + 3y^4 = 2\sin 2t \cos t + 3\cos^4 t$	$\frac{dx}{dt} = 2\cos 2t$
$\frac{\partial z}{\partial y} = x^2 + 12xy^3 = \sin^2 2t + 12 \sin 2t \cos^3 t$	$\frac{dy}{dt} = -\sin t$

$$\frac{dz}{dt} = (2\sin 2t \cos t + 3\cos^4 t)(2\cos 2t) + (\sin^2 2t + 12\sin 2t \cos^3 t)(-\sin t)$$

Evaluate at t=0

$$\left.\frac{dz}{dt}\right|_{t=0} = (0+3)(2) + (\square)(0) = 6$$

Example.

For $w = x^4 y + y^2 z^3$, and $x = se^t, y = s^2 e^{-t}, z = s\sin t$. Find the two partial derivatives $\frac{\partial w}{\partial s}, \frac{\partial w}{\partial t}$.

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\left(\frac{\partial x}{\partial t}\right) + \frac{\partial w}{\partial y}\left(\frac{\partial y}{\partial t}\right) + \frac{\partial w}{\partial z}\left(\frac{\partial z}{\partial t}\right)$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\left(\frac{\partial x}{\partial s}\right) + \frac{\partial w}{\partial y}\left(\frac{\partial y}{\partial s}\right) + \frac{\partial w}{\partial z}\left(\frac{\partial z}{\partial s}\right)$$

$\frac{\partial w}{\partial x} = 4x^3 y = 4(se^t)^3 s^2 e^{-t}$ $= 4s^5 e^{2t}$	$\frac{\partial x}{\partial t} = se^t$	$\frac{\partial x}{\partial s} = e^t$
$\frac{\partial w}{\partial y} = x^4 + 2yz^3$ $= (se^t)^4 + 2(s^2e^{-t})(s\sin t)^3$ $= s^4 e^{4t} + 2s^5 e^{-t}\sin^3 t$	$\frac{\partial y}{\partial t} = -s^2 e^{-t}$	$\frac{\partial y}{\partial s} = 2se^{-t}$
$\frac{\partial w}{\partial z} = 3y^2 z^2$ $= 3(s^2e^{-t})^2(s\sin t)^2$ $= 3s^6 e^{-2t}\sin^2 t$	$\frac{\partial z}{\partial t} = s\cos t$	$\frac{\partial z}{\partial s} = \sin t$

$$\frac{\partial w}{\partial t} = (4s^5e^{2t})(se^t) + (s^4e^{4t} + 2s^5e^{-t}\sin^3 t)(-s^2e^{-t}) + (3s^6e^{-2t}\sin^2 t)(s\cos t)$$

$$\frac{\partial w}{\partial s} = (4s^5e^{2t})(e^t) + (s^4e^{4t} + 2s^5e^{-t}\sin^3 t)(2se^{-t}) + (3s^6e^{-2t}\sin^2 t)(\sin t)$$

In principal, you can check your answer by substituting first and doing worse calculus, and more algebra, but your answers should match eventually.

Implicit Partial Differentiation

Start with a calc I problem with an implicit function of two variables

$$x^3 + y^3 = 6xy$$

We assume that y is a function of x, we just don't know what it is. And so when we take the derivative for y, we use the chain rule, we take the derivative of x normally, and any products of x and y, have to be differentiated with the product rule.

$$3x^2 + 3y^2y' = 6y + 6xy'$$

$$3x^2 + 3y^2\left(\frac{dy}{dx}\right) = 6y + 6x\left(\frac{dy}{dx}\right)$$

Solve for the derivative for y

$$3x^2 + 3y^2y' = 6y + 6xy'$$

$$3x^2 - 6y = 6xy' - 3y^2y'$$
$$3x^2 - 6y = y'(6x - 3y^2)$$

$$y' = \frac{dy}{dx} = \frac{3x^2 - 6y}{6x - 3y^2} = \frac{x^2 - 2y}{2x - y^2}$$

The derivative is a function of both x and y.

In the three-variable case, we're taking a partial derivative and one variable is the variable is the variable we are doing the derivative for (normal variable), one variable is a constant, and then the third variable is the function variable it would require being multiplied by the chain rule. Products will also behave differently: functions with the main variable will need product rules, but function with the other (constant) variable won't.

There is also more than one derivative, so you'll have to do this twice, one for each "independent" variable.

In the case of x,y,z then z is the function variable and x,y are the "independent variables"
In the case of x,y,z,w then w is the function variable and x,y,z are the "independent" variables.

Example. Find the partial derivatives of z for the implicit function $x^2 + y^2 + z^2 + 6xyz = 1$

Find the partial derivative for x first.
X is the variable of the derivative
Y is the constant
Z is the function (assume z=f(x,y))
Don't use z': use $\frac{\partial z}{\partial x}, or\ z_x$

$$2x + 0 + 2zz_x + 6yz + 6xyz_x = 0$$

$$2x + 6yz = -2zz_x - 6xyz_x$$
$$2x + 6yz = z_x(-2z - 6xy)$$

$$z_x = \frac{\partial z}{\partial x} = \frac{2x + 6yz}{-2z - 6xy} = \frac{x + 3yz}{-z - 3xy}$$

Find the partial for y…
X is the constant variable
Y is the variable of the derivative
Z is the function (assume z=f(x,y))
Don't use z': use $\frac{\partial z}{\partial y}$ or z_y

$$0 + 2y + 2zz_y + 6xz + 6xyz_y = 0$$

$$2y + 6xz = -2zz_y - 6xyz_y$$

$$2y + 6xz = -z_y(-2z - 6xy)$$

$$z_y = \frac{\partial z}{\partial y} = \frac{2y + 6xz}{-2z - 6xy} = \frac{y + 3xz}{-z - 3xy}$$

Example.
Find the partial derivatives of z for the implicit function $yz + x \ln y = z^2$

$$yz_x + \ln y = 2zz_x$$

$$\ln y = 2zz_x - yz_x$$

$$\ln y = z_x(2z - y)$$

$$z_x = \frac{\partial z}{\partial x} = \frac{\ln y}{2z - y}$$

$$z + yz_y + \frac{x}{y} = 2zz_y$$

$$z + \frac{x}{y} = 2zz_y - yz_y$$

$$z + \frac{x}{y} = z_y(2z - y)$$

$$z_y = \frac{\partial z}{\partial y} = \frac{\left(z + \frac{x}{y}\right)}{2z - y} = \frac{zy + x}{2zy - y^2}$$

Short-cut formulas for implicit differentiation

Build a function F that has all the terms of little f on one side the equation.

Example.
Find $\frac{dy}{dx}$ for

$$x^3 + y^3 = 6xy$$
$$F(x,y) = x^3 + y^3 - 6xy$$

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}$$

$$F_x = 3x^2 - 6y$$
$$F_y = 3y^2 - 6x$$

$$\frac{dy}{dx} = -\frac{3x^2 - 6y}{3y^2 - 6x}$$

Compare with the previous answer:

$$\frac{dy}{dx} = \frac{3x^2 - 6y}{6x - 3y^2}$$

If we distribute the negative sign through the denominator, we get identical answers.

Consider the 3-variable case:

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = -\frac{F_x}{F_z}$$

$$\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} = -\frac{F_y}{F_z}$$

Find the partial derivatives of z for the implicit function $x^2 + y^2 + z^2 + 6xyz = 1$

$$F(x,y,z) = x^2 + y^2 + z^2 + 6xyz - 1$$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{2x + 6yz}{2z + 6xy} = \frac{2x + 6yz}{-2z - 6xy}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{2y + 6xz}{2z + 6xy} = \frac{2y + 6xz}{-2z - 6xy}$$

Find the partial derivatives of z for the implicit function $yz + x\ln y = z^2$

$$F(x,y) = yz + x\ln y - z^2$$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{\ln y}{y - 2z} = \frac{\ln y}{2z - y}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{\left(z + \frac{x}{y}\right)}{y - 2z} = \frac{\left(z + \frac{x}{y}\right)}{2z - y}$$

Maxima & Minima (Extrema) (14.7/4.7)
We are only going to be considering z=f(x,y) cases

There are essentially 4 possible outcomes of the second derivative test: maximum, minimum, saddle point, or undetermined

Second Partials Test, D-Test

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - \left(f_{xy}\right)^2$$

The value of D (evaluated at a particular critical point) helps determine whether the point is 1) a maximum or a minimum, 2) a saddle point, 3) undetermined

1) Maximum or minimum, D>0
2) Saddle point, D<0
3) Undetermined, D=0

In the case of a maximum or minimum (case 1), look at the concavity of f_{xx} $\left(or\ f_{yy}\right)$ to determine which it is:

$$f_{xx} > 0, minimum$$
$$f_{xx} < 0, maximum$$

Example.
Find and classify the critical points for $f(x,y) = 10x^2y - 5x^2 - 4y^2 - x^4 - 2y^4$

$$f_x = 20xy - 10x - 4x^3$$
$$f_y = 10x^2 - 8y - 8y^3$$

$$f_{xx} = 20y - 10 - 12x^2$$
$$f_{yy} = -8 - 24y^2$$
$$f_{xy} = 20x$$

To find the critical points, set the first derivatives =0

$$20xy - 10x - 4x^3 = 0$$
$$10x^2 - 8y - 8y^3 = 0$$

$$20xy - 10x - 4x^3 = 0$$
$$2x(10y - 5 - 2x^2) = 0$$
$$x = 0, 10y - 5 - 2x^2 = 0$$

Using $x = 0$
Plug into the other equation.

$$-8y - 8y^3 = 0$$
$$-8y(1 + y^2) = 0$$

$$y = 0$$

One critical point is (0,0)

$$10y - 5 - 2x^2 = 0$$
$$10y = 5 + 2x^2$$

$$y = \frac{5 + 2x^2}{10}, x^2 = \frac{10y - 5}{2}$$

$$10x^2 - 8y - 8y^3 = 0$$
$$10\left(\frac{10y - 5}{2}\right) - 8y - 8y^3 = 0$$

$$5(10y - 5) - 8y - 8y^3 = 0$$

$$50y - 25 - 8y - 8y^3 = 0$$

$$-25 + 42y - 8y^3 = 0$$

Find the values for y:

$$y \approx -2.5452, y \approx 0.6468, y \approx 1.8984$$
$$x^2 = \frac{10y - 5}{2}$$

Find corresponding values for x.

$$x^2 = \frac{10(-2.5452) - 5}{2} \approx -15.226 \ldots \textit{not real value for } x$$

$$x^2 = \frac{10(0.6468) - 5}{2} \approx 0.734 \ldots \approx \pm 0.8567 \ldots$$

$$x^2 = \frac{10(1.8984) - 5}{2} \approx 6.992 \ldots \approx \pm 2.6442 \ldots$$

More critical points:

$$(\,0.8567, 0.6468), (-\,0.8567, 0.6468), (2.6442, 1.8984), (\,-\,2.6442, 1.8984)$$

Along with (0,0) from before.

$$f_{xx} = 20y - 10 - 12x^2$$
$$f_{yy} = -8 - 24y^2$$
$$f_{xy} = 20x$$

$$D = (20y - 10 - 12x^2)(-8 - 24y^2) - (20x)^2$$

$$D(0{,}0) = (-10)(-8) - 0^2 = 80\ (maximum, since\ concave\ down)$$

$$D(0.8567, 0.6468) = negative, saddle\ point$$
$$D(-\,0.8567{,}0.6468) = negative, saddle\ point$$
$$D(2.6442{,}1.8984) = positive, maximum$$
$$D(-\,2.6442{,}1.8984) = postive, maximum$$

Next time we'll do another example, and also absolute maxima.
No class tomorrow for Juneteenth.