5/23/2024

Limits in 2 or more variables
Vector-Valued Functions (Domain/sketching)
Vector Fields
Vector-Valued Functions (Derivatives/Integrals)

A note about class times:
When I was assigned to teach this course and made the syllabus, SIS had the correct times in the system: 6-8:15 MTWR. The current listing of 6-7:50 is not correct. It is not equivalent to a full semester lecture times.

If we were meeting in a regular semester for 15 weeks we would meet for around 2 hours (such as 6-7:50) two days per week. We are meeting 4 days per week, so that cuts the number of weeks required in half or 7.5 weeks. We don't have 7.5 weeks, though, we have 6. That leaves us 6 lectures short (1.5 x 4 classes per week), or around 12 hours short. That means we would need 2 additional hours per week. We could meet a 5th day, or spread those hours over the other 4 days at roughly half-an-hour extra per meeting: which puts us around 8:15 instead of 7:50. The number of lecture hours we meet is required for maintaining accreditation, and to make sure we are able to cover all the required material. If we don't maintain the 8:15 projected end time, then I would probably need to make you responsible for learning some material on your own, without being able to lecture on it at all.

Limits in 2 or more variables.

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

If for every number $\varepsilon > 0$ there exists a corresponding $\delta > 0$ such that if (x, y) is in the domain and $\sqrt{(x-a)^2 + (y-b)^2} < \delta$, then $|f(x,y) - L| < \varepsilon$

Example.

$$\lim_{(x,y)\to(1,2)} 5x^3 - x^2y^2 = 5(1)^3 - (1)^2(2)^2 = 5 - 4 = 1$$

Example.

$$\lim_{(x,y)\to(0,0)} \frac{x-y}{\sqrt{x-y}} =$$

Let $x - y = u$

$$\lim_{u\to 0} \frac{u}{\sqrt{u}} = \lim_{u\to 0} \sqrt{u} = 0$$

Example.

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} =$$

Let $u = x^2 + y^2$, or $r^2 = x^2 + y^2$

$$\lim_{u\to 0} \frac{\sin(u)}{u} = \lim_{u\to 0} \frac{\cos(u)}{1} = 1$$

L'Hopital's rule does not apply to two variable cases, only one-variable cases.

Example.

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

The difficult with multivariable limits is that we need to approach the origin from every possible direction to prove that the limit is the same all the time.

Try various paths to the origin:

1. $x = 0$

$$\lim_{y\to 0} \frac{0y}{0+y^2} = \lim_{y\to 0} \frac{0}{y^2} = \lim_{y\to 0} 0 = 0$$

2. $y = 0$

$$\lim_{x\to 0} \frac{x(0)}{x^2+0} = \lim_{x\to 0} \frac{0}{x^2} = \lim_{x\to 0} 0 = 0$$

3. $y = x$

$$\lim_{x\to 0} \frac{x(x)}{x^2+x^2} = \lim_{x\to 0} \frac{x^2}{2x^2} = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2}$$

The limit is DNE

Let's try polar coordinates:

$$\lim_{r\to 0} \frac{r\cos\theta \; r\sin\theta}{r^2} = \lim_{r\to 0} \frac{r^2\cos\theta\sin\theta}{r^2} = \lim_{r\to 0} \frac{\cos\theta\sin\theta}{1} = \cos\theta\sin\theta$$

The value of the expression will change as the value of θ changes.

Since θ is left behind after arriving at the origin, the limit has different values depending on the path one takes, and therefore the limit does not exist (DNE)

Choosing the critical path:
Choose a path to origin that will allow the denominator to simplify as much as possible.

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$$

The idea is to make a substitution so that afterwards, the powers in the denominator are the same.

$$x^2 = y^4$$
$$x = y^2$$

Substitute

$$x = ky^2$$

$$\lim_{y\to 0} \frac{ky^2y^2}{(ky^2)^2+y^4} = \lim_{y\to 0} \frac{ky^4}{k^2y^4+y^4} = \lim_{y\to 0} \frac{ky^4}{y^4(k^2+1)} = \lim_{y\to 0} \frac{k}{(k^2+1)} = \frac{k}{(k^2+1)}$$

This limit does not exist.

Consider the previous problem:

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

The critical path is $y = kx$

$$\lim_{x\to 0} \frac{x(kx)}{x^2+(kx)^2} = \lim_{x\to 0} \frac{kx^2}{x^2+k^2x^2} = \lim_{x\to 0} \frac{kx^2}{x^2(1+k^2)} = \lim_{x\to 0} \frac{k}{(1+k^2)} = \frac{k}{(1+k^2)}$$

This is also DNE and is consistent with the polar substitution that we did.

$$\lim_{(x,y)\to(0,0)} \frac{x^4-y^4}{x^2+y^2}$$

Polar coordinates

$$\lim_{r\to 0} \frac{r^4\cos^4\theta - r^4\sin^4\theta}{r^2} = \lim_{r\to 0} \frac{r^4(\cos^4\theta - \sin^4\theta)}{r^2} = \lim_{r\to 0} r^2(\cos^4\theta - \sin^4\theta) = 0$$

In a three-variable problem:
One way is to reduce the problem to the two-variable case, and then apply our strategies for the two-variable.

May also be able to apply spherical coordinates

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^2+y^2+z^2}$$

$$\lim_{\rho\to 0} \frac{\rho\cos\theta\sin\phi\,\rho\sin\theta\sin\phi\,\rho\cos\phi}{\rho^2} = \lim_{\rho\to 0} \frac{\rho^3\cos\theta\sin\phi\sin\theta\sin\phi\cos\phi}{\rho^2} =$$

$$\lim_{\rho\to 0} \rho\cos\theta\sin\phi\sin\theta\sin\phi\cos\phi = 0$$

Vector-Valued Functions (13.1/3.1)
Introduced these when we did lines in space

$$\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$$

Domain:
Can express in traditional interval terms in just one variable t.
Find the domain of each component of the vector separately and then find the intervals where all the components are defined.

$$\vec{r}(t) = \langle \frac{t}{t-2}, \sqrt{t}, \sin(t)\rangle$$

x-direction: $t \neq 2$
y-direction: $t \geq 0$
z-direction: t is any real number

Domain: $[0,2) \cup (2, \infty)$

It may be helpful to graph the intervals on one or more number lines to help figure out where all of them are true.

Sketching curves in three dimensions

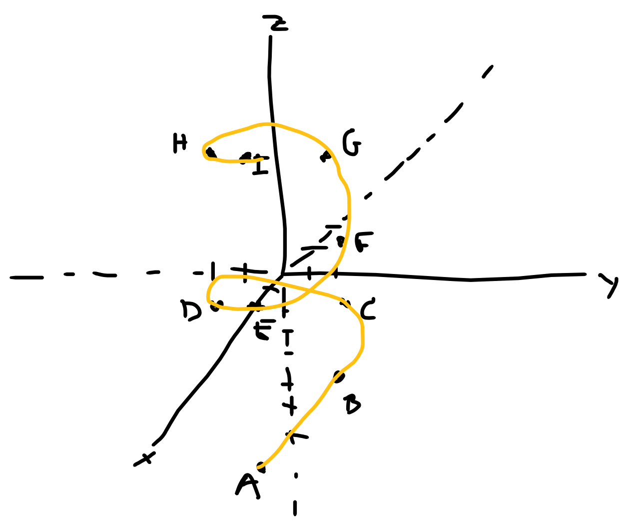
$$\vec{r}(t) = \langle 2\cos t, 2\sin t, t \rangle$$

A helix is essentially a slinky.

Make sure you graph (for a helix) at least two turns around the circle – 0 to 4pi or -2pi to 2pi

Build a table of values, plot the individual points, connect the dots in order

t	x	y	z	
-2π	2	0	-2π	A
$-\frac{3\pi}{2}$	0	2	$-\frac{3\pi}{2}$	B
$-\pi$	-2	0	$-\pi$	
$-\frac{\pi}{2}$	0	-2	$-\frac{\pi}{2}$	
0	2	0	0	
$\frac{\pi}{2}$	0	2	$\frac{\pi}{2}$	
π	-2	0	π	
$\frac{3\pi}{2}$	0	-2	$\frac{3\pi}{2}$	
2π	2	0	2π	

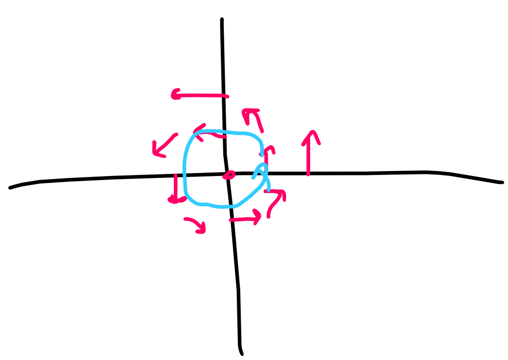


Vector Fields (16.1/6.1)
Has a vector associated with every point in the plane or in space.

$$\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$$

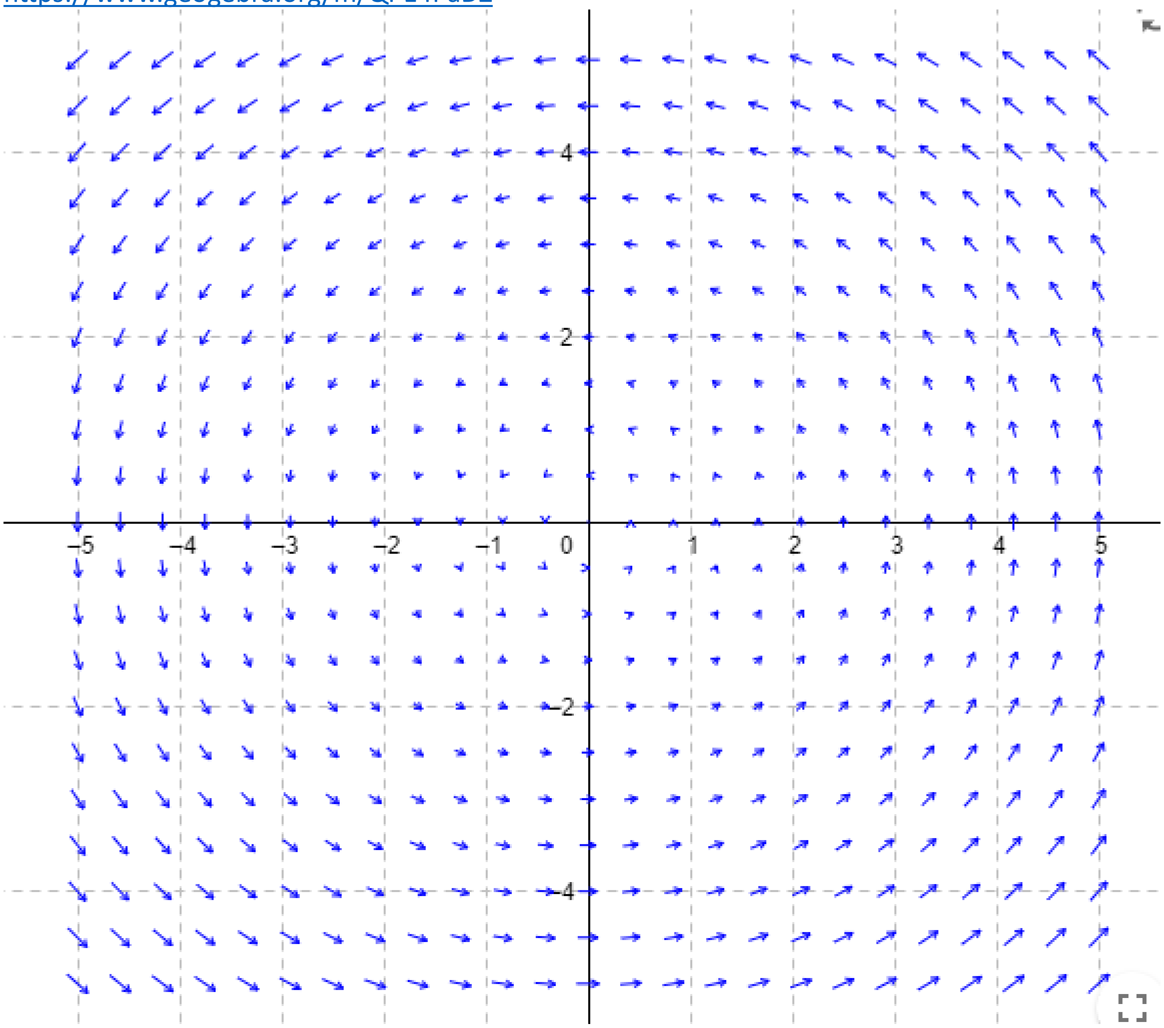
$$\vec{F}(x,y) = \langle -y, x \rangle$$

x	y	$\langle -y, x \rangle$
0	0	$\langle 0,0 \rangle$
1	0	$\langle 0,1 \rangle$
0	1	$\langle -1,0 \rangle$
-1	0	$\langle 0,-1 \rangle$
0	-1	$\langle 1,0 \rangle$
1	1	$\langle -1,1 \rangle$
-1	1	$\langle -1,-1 \rangle$
-1	-1	$\langle 1,-1 \rangle$
1	-1	$\langle 1,1 \rangle$



2	0	$\langle 0,2\rangle$
0	2	$\langle -2,0\rangle$

https://www.geogebra.org/m/QPE4PaDZ



Calculus on Vector-Valued Functions (like the helix)

When you take derivatives (or integrals) on vector-valued functions, you do them component by component.

$$\vec{r}(t) = \langle 2\cos t\,, 2\sin t\,, t\rangle$$
$$\vec{r}\,'(t) = \langle -2\sin t\,, 2\cos t\,, 1\rangle$$
$$\int \vec{r}(t)dt = \langle 2\sin t + C_1, -2\cos t + C_2, \frac{t^2}{2} + C_3\rangle$$