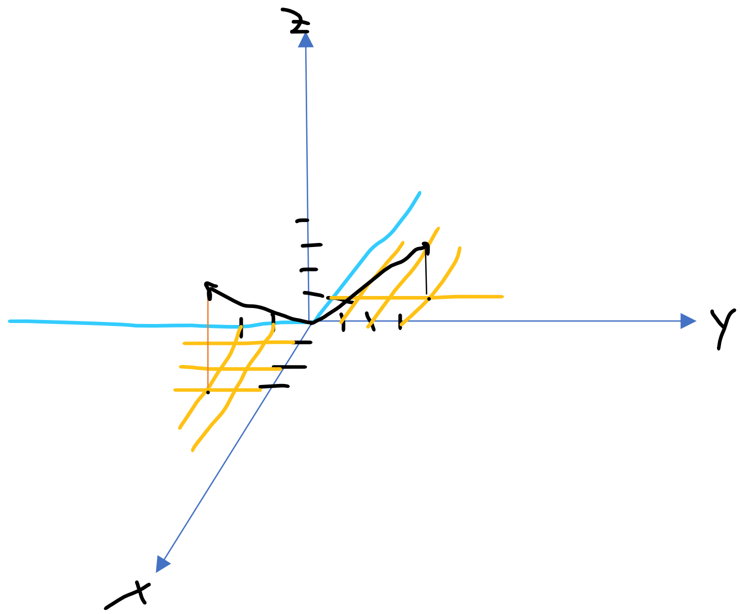
5/21/2024

Dot Product & Cross Product



Vector $\langle 3,-2,4\rangle$, $\langle -1,3,2\rangle$

Dot Product (12.3/2.3)
Also sometimes called the scalar product or inner product

$$\vec{u}\cdot\vec{v} = u_1v_1 + u_2v_2 + u_3v_3$$

$\vec{u} = \langle 3,-2,4\rangle$, $\vec{v} = \langle -1,3,2\rangle$

$$\vec{u}\cdot\vec{v} = 3(-1) + (-2)(3) + 4(2) = -3 - 6 + 8 = -1$$

$$\vec{u}\cdot\vec{v} = \|\vec{u}\|\|\vec{v}\|\cos\theta$$

θ is the angle between the vectors.

Magnitude of the first force was 20 pounds, and at an angle of 45-degrees (positive), and the second force was 16 pounds and at an angle of negative 30-degrees.
Here the dot product would be $20(16)cos(75°) = 82.822$ …

Find the angle between two vectors:

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}$$

$$\cos\theta = -\frac{1}{\sqrt{29}\sqrt{14}}$$

$$\theta = \cos^{-1}\left(-\frac{1}{\sqrt{29}\sqrt{14}}\right) = 92.8°, 1.62\ radians$$

The sign of the dot product tells us whether the angle between the vectors is acute or obtuse. If the dot product is positive, then the angle is acute, and if it is negative, then the angle is obtuse. If the dot product is 0, then the vectors are perpendicular (orthogonal), the angle between them is 90-degrees or $\frac{\pi}{2}$.

If I want to find a vector that is perpendicular to another vector, then I can use the dot product and set it equal to 0 to obtain a condition on the new vector.

$\langle 1, -2, 3\rangle$, find a vector perpendicular to this one

$$\langle 1, -2, 3\rangle \cdot \langle a, b, c\rangle = a - 2b + 3c = 0$$

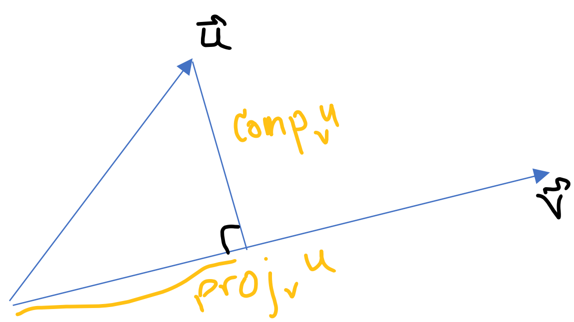
Suppose I let b=1, c=1, then $a - 2(1) + 3(1) = 0 \rightarrow a + 1 = 0, a = -1$

$$\langle 1, -2, 3\rangle \cdot \langle -1, 1, 1\rangle = -1 - 2 + 3 = 0$$

Projections

$$proj_{\vec{v}}(\vec{u}) = \left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2}\right)\vec{v}$$

The projection of the vector u onto the vector v



The project is a scalar multiple of v (a vector in the same direction as v)

$$comp_{\vec{v}}(\vec{u}) = \vec{u} - proj_{\vec{v}}(\vec{u})$$

$\vec{u} = \langle 3, -2, 4\rangle$, $\vec{v} = \langle -1, 3, 2\rangle$

$$proj_{\vec{v}}(\vec{u}) = \left(\frac{-1}{14}\right)\langle -1, 3, 2\rangle = \langle \frac{1}{14}, -\frac{3}{14}, -\frac{1}{7}\rangle$$

$$comp_{\vec{v}}(\vec{u}) = \langle 3, -2, 4\rangle - \langle \frac{1}{14}, -\frac{3}{14}, -\frac{1}{7}\rangle = \langle \frac{41}{14}, -\frac{25}{14}, \frac{29}{7}\rangle$$

The work done by a force applied at an angle to the direction of motion

$$W = \vec{F} \cdot \vec{d}$$

Suppose a force is applied to an object with the force vector $\vec{F} = \langle 3, 4, 5\rangle$ and it moves an object from a point in space at P(2,1,0) to Q(4,6,2).

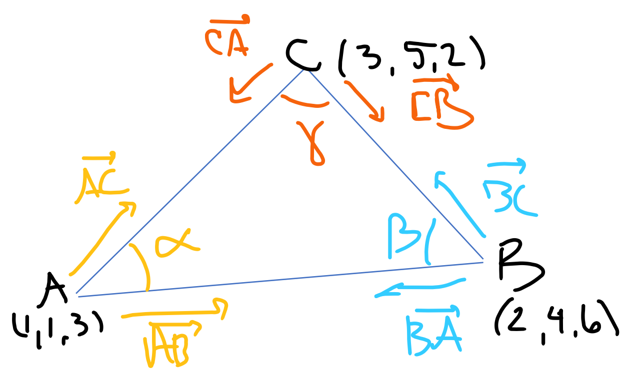
The direction vector: $\langle 4-2, 6-1, 2-0\rangle = \langle 2,5,2\rangle = \overrightarrow{PQ} = \vec{d}$

$$W = \langle 3, 4, 5\rangle \cdot \langle 2,5,2\rangle = 6 + 20 + 10 = 36\, units\ of\ Work$$

Can use it to determine the type of triangle you have based on points in space.

Triangle defined by $(1,1,3), (2,4,6), (3,5,2)$

What kind of triangle is this? Equilateral? Isosceles? Scalene? Right triangle? Acute triangle? Obtuse triangle?



$$\overrightarrow{AC} = \langle 3-1, 5-1, 2-3\rangle = \langle 2,4,-1\rangle$$
$$\overrightarrow{AB} = \langle 2-1, 4-1, 6-3\rangle = \langle 1,3,3\rangle$$
$$\overrightarrow{AC} \cdot \overrightarrow{AB} = 2 + 12 - 3 = 11\ (acute)$$

$$\overrightarrow{BC} = \langle 3-2, 5-4, 2-6\rangle = \langle 1,1,-4\rangle$$
$$\overrightarrow{BA} = -\overrightarrow{AB} = \langle -1,-3,-3\rangle$$
$$\overrightarrow{BC} \cdot \overrightarrow{BA} = -1 - 3 + 12 = 8\ (acute)$$

$$\overrightarrow{CA} = -\overrightarrow{AC} = \langle -2,-4,1\rangle$$
$$\overrightarrow{CB} = -\overrightarrow{BC} = \langle -1,-1,4\rangle$$
$$\overrightarrow{CA} \cdot \overrightarrow{CB} = 2 + 4 + 4 = 10\ (acute)$$

$$\alpha = \cos^{-1}\left(\frac{11}{\sqrt{21}\sqrt{19}}\right) \approx 0.9876\, radians$$
$$\beta = \cos^{-1}\left(\frac{8}{\sqrt{18}\sqrt{19}}\right) \approx 1.1234 \ldots radians$$

$$\gamma = \cos^{-1}\left(\frac{10}{\sqrt{21}\sqrt{18}}\right) \approx 1.030 \ldots radians$$

Alternatively, you can find the lengths of the sides using the distance formula instead of finding the angles for the equilateral/isosceles/scalene step.

Cross Product (12.4/2.4)
Another kind of vector product, sometimes called the outer product
The result of a cross product is another vector
We are going to define this specifically for three dimensions.

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \hat{\imath} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \hat{k} =$$

$$(u_2 v_3 - v_2 u_3)\hat{\imath} - (u_1 v_3 - v_1 u_3)\hat{\jmath} + (u_1 v_2 - v_1 u_2)\hat{k}$$

Find the cross product of $\vec{u} \times \vec{v}$ for $\vec{u} = \langle 3, -2, 4\rangle$, $\vec{v} = \langle -1, 3, 2\rangle$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & -2 & 4 \\ -1 & 3 & 2 \end{vmatrix} = \begin{vmatrix} -2 & 4 \\ 3 & 2 \end{vmatrix} \hat{\imath} - \begin{vmatrix} 3 & 4 \\ -1 & 2 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} 3 & -2 \\ -1 & 3 \end{vmatrix} \hat{k} =$$

$$\langle (-4) - 12, -(6+4), 9 - 2\rangle = \langle -16, -10, 7\rangle$$

The cross product of two vectors is perpendicular to the plane that contains both vectors (it's perpendicular to both vectors).

If you switch the order of the cross product you get a negative:

$$\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$$

A short-cut for 3D vector cross products only (don't apply to higher dimensional determinants)

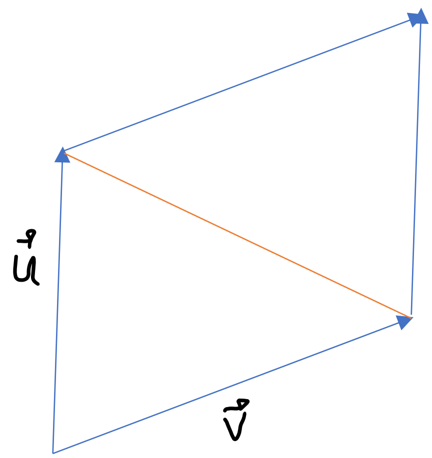
$$\left.\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & -2 & 4 \\ -1 & 3 & 2 \end{vmatrix} \begin{matrix} \hat{\imath} & \hat{\jmath} \\ 3 & -2 \\ -1 & 3 \end{matrix}\right. =$$

$$\langle -4 - 12, -4 - 6, 9 - 2\rangle = \langle -16, -10, 7\rangle$$

If you dot a vector with itself, you get the magnitude squared $\vec{u} \cdot \vec{u} = \|\vec{u}\|^2$
If you cross a vector with itself, you get the 0 vector $\vec{u} \times \vec{u} = \langle 0,0,0\rangle = \vec{0}$

$$\|\vec{u} \times \vec{v}\| = \|\vec{u}\|\|\vec{v}\| \sin\theta$$

We can use the cross product to find the area of a parallelogram, when the parallelogram has sides defined by two vectors (coming out of the same point).



$$\vec{u} = \langle -3,1,-7\rangle, \vec{v} = \langle 0,-5,-5\rangle$$

$$\vec{u}\times\vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -3 & 1 & -7 \\ 0 & -5 & -5 \end{vmatrix} = \langle -5-35, -(15), 15\rangle = \langle -40,-15,15\rangle$$

Area is the magnitude: $A = \sqrt{1600+225+225} = \sqrt{2050} = 5\sqrt{82} \approx 45.3$

A triangle is have the area of a parallelogram, I can also use the cross product to find the area of a triangle, but divide the magnitude by 2.

Triple Scalar Product

$$\vec{u}\cdot(\vec{v}\times\vec{w})$$

$$= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \left(\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix}\right)u_1 - \left(\begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix}\right)u_2 + \left(\begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}\right)u_3 =$$

$$(v_2w_3 - w_2v_3)u_1 - (v_1w_3 - w_1v_3)u_2 + (v_1w_2 - w_1v_2)u_3 =$$

Find the triple scalar product of $\langle 1,4,-7\rangle, \langle 2,-1,4\rangle, \langle 0,-9,18\rangle$

$$\begin{vmatrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ 0 & -9 & 18 \end{vmatrix} = 1(-18+36) - 4(36-0) + (-7)(-18-0) = 18 - 144 + 126 = 0$$

If the triple scalar product is 0, that means that the vectors all lie in the same plane (coplanar).

Three vectors can be used to find the bounds of a parallelepiped (slanty box). The triple scalar (the absolute value of it) gives the volume of the parallelepiped defined by three vectors (from one corner).

Next time: Equations of lines and planes in 3D
Applications for dot product and cross product: distance between a line and a point, or a plane and a point.