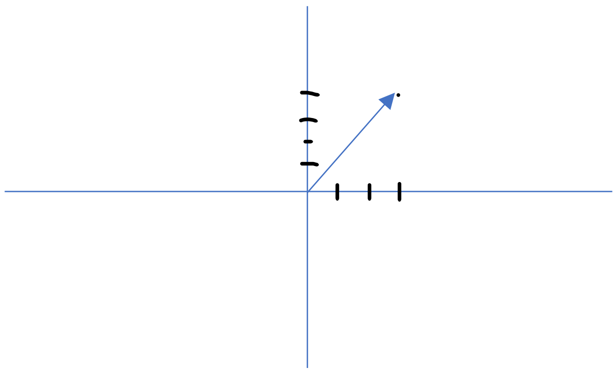
5/20/2024

Introduction to the course
Vectors

Vectors in 2D
Vector notation:

$$\vec{v} = \langle 3,4 \rangle = 3\hat{\imath} + 4\hat{\jmath} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$



Vectors of magnitude and direction.
Magnitude of a vector is the length of the vector:

$$\|\vec{v}\| = \sqrt{a^2 + b^2} = \sqrt{v_1^2 + v_2^2} = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$$

Direction:
In two dimensions, we can find the angle relative to the positive x-axis.

$$\tan^{-1}\left(\frac{y}{x}\right) = \theta$$

$$\tan^{-1}\left(\frac{4}{3}\right) = 0.927295 \ldots, or\ 53.1°$$

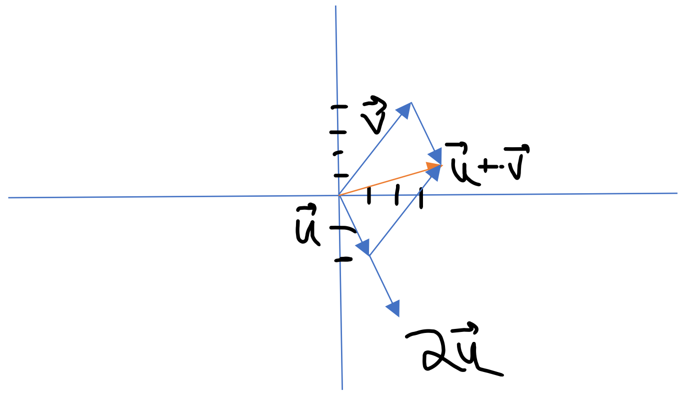
A unit vector can also be used to represent direction in 2 or three dimensions:
A unit vector is a vector of length 1, and we can find a unit vector from any vector by dividing the original vector by its magnitude.

A unit vector in the direction of $\vec{v} = \frac{\langle 3,4 \rangle}{5} = \langle \frac{3}{5}, \frac{4}{5} \rangle$

Add vectors we add them component by component.

$$\vec{u} = \langle 1, -2 \rangle, \vec{v} = \langle 3,4 \rangle$$

$$\vec{u} + \vec{v} = \langle 4, 2 \rangle$$



Parallelogram rule: the sum of two vectors is the diagonal of the parallelogram formed by the two vectors.

Scale vectors: multiply through by a constant, apply to each term.

$$2\vec{u} = 2\langle 1, -2\rangle = \langle 2, -4\rangle$$

Combinations of scaling and adding (subtracting)

$$3\vec{u} - 2\vec{v} = 3\langle 1, -2\rangle - 2\langle 3,4\rangle = \langle 3, -6\rangle + \langle -6, -8\rangle = \langle -3, -14\rangle$$

In 3D:

$$\vec{v} = \langle 1,3,5\rangle = \hat{\imath} + 3\hat{\jmath} + 5\hat{k}$$

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{1^2 + 3^2 + 5^2} = \sqrt{1 + 9 + 25} = \sqrt{35}$$

Direction would be the unit vector:

$$\hat{v} = \frac{\vec{v}}{\|\vec{v}\|} = \langle \frac{1}{\sqrt{35}}, \frac{3}{\sqrt{35}}, \frac{5}{\sqrt{35}}\rangle$$

Add and scale vectors, they also work the same way:

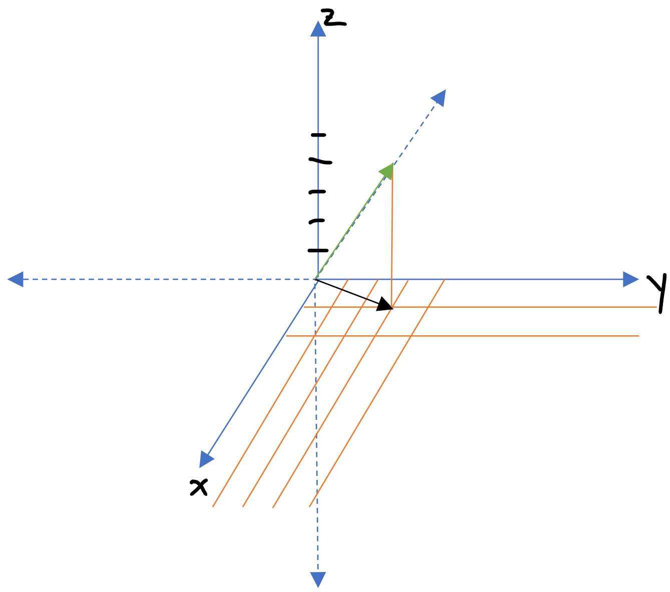
$$\vec{u} = \langle -3,2,-1\rangle, \vec{v} = \langle 1,3,5\rangle$$

Find

$$2\vec{u} - 3\vec{v} = 2\langle -3,2,-1\rangle - 3\langle 1,3,5\rangle = \langle -6,4,-2\rangle + \langle -3,-9,-15\rangle = \langle -9,-5,-17\rangle$$

Right-hand rule vs. left-hand rule

In math we use the right-hand rule:



Direction cosines:

$$\alpha, \beta, \gamma$$

The angles that a vector in three dimensions makes with each axis.

$$\cos(\alpha) = \frac{v_1}{||\vec{v}||}, \cos(\beta) = \frac{v_2}{||\vec{v}||}, \cos(\gamma) = \frac{v_3}{||\vec{v}||}$$

$$\vec{v} = \langle 1,3,5 \rangle$$

$$\cos(\alpha) = \frac{1}{\sqrt{35}}, \cos(\beta) = \frac{3}{\sqrt{35}}, \cos(\gamma) = \frac{5}{\sqrt{35}}$$

$$\alpha = 1.40, \beta = 1.039, \gamma = 0.564$$

Some simple 3D representations extended from 2D:

$x = 3$ in 2D is a line, but in 3D this is a plane.
To get a line in 3D, you need the intersection of two planes

If we have an equation with two variables, we can plot the curve in the plane, but then the third dimension is free.

For example:

$$x^2 + y^2 = 4$$

In 2d: this is a circle of radius 2
But in 3D: z-coordinate is free: so this is a cylinder (circular)

A sphere in 3D:

$$(x - h)^2 + (y - h)^2 + (z - l)^2 = \rho^2$$

Center of the sphere is at (h, k, l) with radius ρ.

Distance formula in 3D:

$$d = \sqrt{a^2 + b^2 + c^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Midpoint formula

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

Ball:

$$(x - h)^2 + (y - h)^2 + (z - l)^2 \le \rho^2$$

Connect our two-dimensional representations to polar coordinates.

$$\vec{v} = \langle x, y \rangle$$
$$\|\vec{v}\| = r = \sqrt{x^2 + y^2}$$
$$\tan^{-1}\left(\frac{y}{x}\right) = \theta$$
$$x = r\cos(\theta) = \|\vec{v}\|\cos(\theta), y = r\sin(\theta) = \|\vec{v}\|\sin(\theta)$$

Suppose I have two forces being applied to an object with one force being applied at an angle of 45-degree with the horizontal (positive x-axis) and has a magnitude of 20 pounds, and a second force applied at an angle of -30-degrees with the horizontal (relative to the positive x-axis) with a magnitude of 16 pounds. What is the resulting force?

$$\vec{F}_1 = \langle 20\cos(45°), 20\sin(45°)\rangle = \langle 10\sqrt{2}, 10\sqrt{2}\rangle$$

$$\vec{F}_2 = \langle 16\cos(-30°), 16\sin(-30°)\rangle = \langle 16\left(\frac{\sqrt{3}}{2}\right), 16\left(-\frac{1}{2}\right)\rangle = \langle 8\sqrt{3}, -8\rangle$$

$$\vec{F}_{total} = \langle 10\sqrt{2} + 8\sqrt{3}, 10\sqrt{2} - 8\rangle = \langle 27.9985 \ldots, 6.142135 \ldots\rangle$$

$$\left\|\vec{F}_{total}\right\| = \sqrt{27.9985^2 + 6.142135^2} \approx 28.66 \ldots pounds$$

$$\tan^{-1}\left(\frac{6.142135}{27.9985}\right) = 12.4°$$

Dot Products and Cross Products next sequence.