7/6/2023

Antiderivatives
(more hyperbolic trig functions)
Approximating area under a curve, definite integrals

Antiderivatives

Antiderivatives are the inverse operation of derivatives (of differentiation). If start with a function which I'll call $f(x)$ and I say that this is the derivative of some other function $F(x)$. We want to be able to find $F(x)$. Then if we take the derivative of $F(x)$ we should be able to get back to $f(x)$.

Derivative Rule	Antiderivative Rule
$\frac{d}{dx}[x^n] = nx^{n-1}$	$\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$
$\frac{d}{dx}[c] = 0$	$\int 0 dx = C$
$\frac{d}{dx}[x] = 1$	$\int k dx = kx + C$
$\frac{d}{dx}[\sin x] = \cos x$	$\int \cos x \, dx = \sin x + C$
$\frac{d}{dx}[\cos x] = -\sin x$	$\int \sin x \, dx = -\cos x + C$
$\frac{d}{dx}[\tan x] = \sec^2 x$	$\int \sec^2 x \, dx = \tan x + C$
$\frac{d}{dx}[\cot x] = -\csc^2 x$	$\int \csc^2 x \, dx = -\cot x + C$
$\frac{d}{dx}[\sec x] = \sec x \tan x$	$\int \sec x \tan x \, dx = \sec x + C$
$\frac{d}{dx}[\csc x] = -\csc x \cot x$	$\int \csc x \cot x \, dx = -\csc x + C$
$\frac{d}{dx}[e^x] = e^x$	$\int e^x dx = e^x + C$
$\frac{d}{dx}[\ln x] = \frac{1}{x}$	$\int \frac{1}{x} dx = \ln \lvert x \rvert + C$
$\frac{d}{dx}[a^x] = (\ln a) a^x$	$\int a^x \, dx = \frac{a^x}{\ln a} + C$
$\frac{d}{dx}[\log_a x] = \frac{1}{x \ln a}$	$\int \frac{1}{x \ln a} dx = \frac{1}{\ln a} \int \frac{1}{x} dx = \frac{1}{\ln a} (\ln \lvert x \rvert) + C$ $= \log_a \lvert x \rvert + C$
$\frac{d}{dx}[\arcsin x] = \frac{d}{dx}[\sin^{-1} x] = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + C$
$\frac{d}{dx}[\arccos x] = \frac{d}{dx}[cos^{-1} x] = \frac{-1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}} dx = -\arccos x + C$
$\frac{d}{dx}[\arctan x] = \frac{d}{dx}[\tan^{-1} x] = \frac{1}{1+x^2}$	$\int \frac{1}{1+x^2} dx = \arctan x + C$

$\frac{d}{dx}[\operatorname{arccot} x] = \frac{d}{dx}[cot^{-1} x] = \frac{-1}{1+x^2}$	$\int \frac{1}{1+x^2} dx = -\operatorname{arccot} x + C$
$\frac{d}{dx}[\operatorname{arcsec} x] = \frac{d}{dx}[\sec^{-1} x] = \frac{1}{\|x\|\sqrt{x^2-1}}$	$\int \frac{1}{\|x\|\sqrt{x^2-1}} dx = \operatorname{arcsec}\|x\| + C$
$\frac{d}{dx}[\operatorname{arccsc} x] = \frac{d}{dx}[\csc^{-1} x] = \frac{-1}{\|x\|\sqrt{x^2-1}}$	$\int \frac{1}{\|x\|\sqrt{x^2-1}} dx = -\operatorname{arccsc}\|x\| + C$
$\frac{d}{dx}[\sinh x] = \cosh x$	$\int \cosh x\, dx = \sinh x + C$
$\frac{d}{dx}[\cosh x] = \sinh x$	$\int \sinh x\, dx = \cosh x + C$
$\frac{d}{dx}[\tanh x] = \operatorname{sech}^2 x$	$\int \operatorname{sech}^2 x\, dx = \tanh x + C$
$\frac{d}{dx}[\coth x] = -\operatorname{csch}^2 x$	$\int \operatorname{csch}^2 x\, dx = -\coth x + C$
$\frac{d}{dx}[\operatorname{sech} x] = -\operatorname{sech} x \tanh x$	$\int \operatorname{sech} x \tanh x\, dx = -\operatorname{sech} x + C$
$\frac{d}{dx}[\operatorname{csch} x] = -\operatorname{csch} x \coth x$	$\int \operatorname{csch} x \coth x\, dx = -\operatorname{csch} x + C$

$$\int x^3 dx = \frac{x^4}{4} + C$$

$$f(x) = x^3, F(x) = \frac{1}{4}x^4 + C$$

$$\frac{d}{dx}[F(x)] = \frac{d}{dx}\left[\frac{1}{4}x^4 + C\right] = \frac{1}{4}\frac{d}{dx}[x^4] + \frac{d}{dx}[C] = \frac{1}{4}(4x^3) + 0 = x^3$$

$$\int \sqrt{x} dx = \int x^{1/2} dx = \frac{2}{3}x^{3/2} + C$$

General properties of antiderivatives mimic the properties of derivatives:

$$\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$$

$$\int kf(x)dx = k\int f(x)dx$$

$$\int \frac{3x^2+2}{x^2} dx = \int \left(\frac{3x^2}{x^2} + \frac{2}{x^2}\right) dx = \int (3 + 2x^{-2})dx = 3x + \frac{2x^{-1}}{-1} + C = 3x - \frac{2}{x} + C$$

$$\int (\cos x + \sec^2 x + 4x)dx = \sin x + \tan x + \frac{4x^2}{2} + C = \sin x + \tan x + 2x^2 + C$$

$$\int e^x - x^{\pi} + \sinh x\, dx = e^x - \frac{x^{\pi+1}}{\pi+1} + \cosh x + C$$

$$\int \frac{1}{2x} - \frac{1}{x^2} + \frac{1}{\sqrt{x}}\,dx = \int \left(\frac{1}{2}\right)\left(\frac{1}{x}\right) - x^{-2} + x^{-\frac{1}{2}}dx = \frac{1}{2}\ln|x| - \frac{x^{-1}}{-1} + x^{\frac{1}{2}}(2) + C$$

$$= \frac{1}{2}\ln|x| + \frac{1}{x} + 2\sqrt{x} + C$$

Some of the problems in this section of the textbook will give you information about the function at a particular point. $F(0) = 1$ (an initial condition) so that you can find the constant.

Suppose that $f(x) = \sqrt{x}$. Find $F(x)$ and find the constant of integration if the initial condition is $F(0) = 1$.

To $F(x)$, we would integrate (find the antiderivative). $F(x) = \frac{2}{3}x^{3/2} + C$.

$$\frac{2}{3}(0)^{\frac{3}{2}} + C = 1 \rightarrow C = 1$$

$$F(x) = \frac{2}{3}x^{3/2} + 1$$

Approximating Area under a Curve

Sigma notation

$$\sum_{i=1}^{10} i^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2 + 10^2$$

$$\sum_{k=1}^{4}\left(\frac{1}{k} + k\right) = \left(\frac{1}{1} + 1\right) + \left(\frac{1}{2} + 2\right) + \left(\frac{1}{3} + 3\right) + \left(\frac{1}{4} + 4\right)$$

$$\sum_{k=1}^{n}\left(\frac{1}{k} + k\right) = \left(\frac{1}{1} + 1\right) + \left(\frac{1}{2} + 2\right) + \left(\frac{1}{3} + 3\right) + \cdots + \left(\frac{1}{n} + n\right)$$

$$\sum_{k=1}^{n}\left(\frac{1}{k} + k\right) = \sum_{k=1}^{n}\frac{1}{k} + \sum_{k=1}^{n}k$$

$$\sum_{i=1}^{5} 6i = 6(1) + 6(2) + 6(3) + 6(4) + 6(5) = 6(1 + 2 + 3 + 4 + 5) = 6\sum_{i=1}^{5} i$$

$$\sum_{j=12}^{20} j^2 = \sum_{j=1}^{20} j^2 - \sum_{j=1}^{11} j^2$$

$$\sum_{j=m}^{n} j^2 = \sum_{j=1}^{n} j^2 - \sum_{j=1}^{m-1} j^2$$

Summation formulas for powers of i

$$\sum_{i=1}^{n} c = c\sum_{i=1}^{n} 1 = cn$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

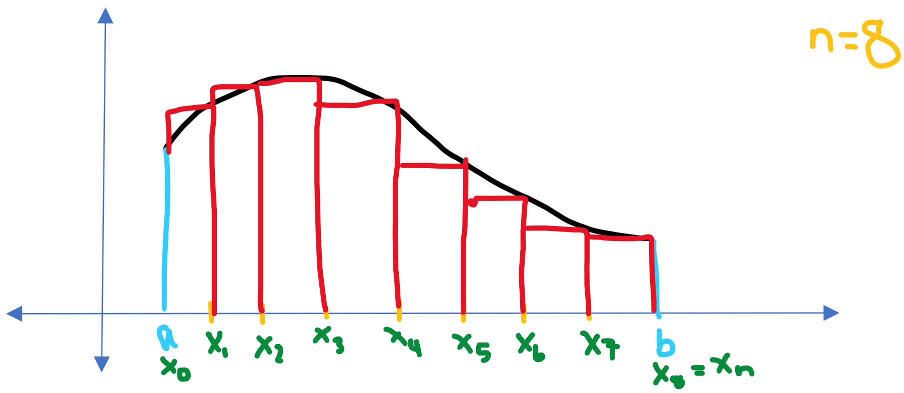
$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

Evaluate $\sum_{i=1}^{16}(2+i)$.

$$\sum_{i=1}^{16}(2+i) = \sum_{i=1}^{16} 2 + \sum_{i=1}^{16} i = 2(16) + \frac{16(17)}{2} = 32 + 136 = 168$$

We would like to approximate the area under a random curve.
If the area is a common "geometric" area like a triangle, a trapezoid, a circle (part of a circle), rectangle, etc. then we don't need calculus to find the area. But what if the area boundary is defined by a parabola instead of a straight line? Or a sine function? Or a log function?

Starting with an approximation: we are going divide up the region where we are calculating the area into a finite set of partitions (typically these are equal size partitions, but they are not strictly required to be equal sizes).



The right-hand rule (use the height of the function on the right edge of the interval as the estimate of the height of the rectangle in that section)
The left-hand rule (use the height of the function on the left edge of the interval as the estimate of the height of the rectangle in that section)
A midpoint rule (use the height of the function at the midpoint of the subinterval as the estimate for the height of the rectangle in that section)

Upper estimate (use the higher end of the function—overestimating)
Lower estimate (uses the lower end of the function—underestimating)

The width of the rectangles are given by Δx_i, but most of the time they will all just be the same, so we use $\Delta x = \frac{b-a}{n}$.

The height of the function is the height of the rectangle $f(x_i)$ (right hand rule)
$f(x_{i-1})$ is for the left-hand rule.

What is the area of each rectangle? Base times height: $f(x_i)\Delta x$

The estimate for the area under the curve is the sum of all of our rectangles:

$$A \approx \sum_{i=1}^{n} f(x_i)\Delta x$$

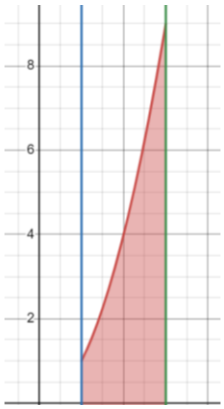
Eventually, we actually want to "true" area:

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

The area under a curve, notationally, is equivalent to a definite integral.
Antiderivatives are indefinite integrals: have no limits, the produce functions.
Definite integrals look very similar, but have limits (endpoints) and produce numbers.

$$A = \int_a^b f(x)dx$$

Example.
Estimate the area under the curve $f(x) = x^2$ on the interval [1,3], n=4.
Right-hand rule.



$$\Delta x = \frac{b-a}{n} = \frac{3-1}{4} = \frac{2}{4} = \frac{1}{2}$$

Partition the interval:

$$\{1, 1.5, 2, 2.5, 3\}$$
$$\{x_0, x_1, x_2, x_3, x_4\}$$

For right-hand rule: use the height of the function at x_1, x_2, x_3, x_4 (skip the first value)
For the left-hand rule: use the height of the function at x_0, x_1, x_2, x_3 (skip the last value)
For the mid-point rule: average consecutive values: $\frac{x_i+x_{i+1}}{2}$ = $\{1.25, 1.75, 2.25, 2.75\}$

Right-hand rule: first rectangle height is $f(x_1) = (1.5)^2 = 2.25$
Second rectangle height is $f(x_2) = (2)^2 = 4$
Third rectangle height is $f(x_3) = (2.5)^2 = 6.25$
Fourth rectangle height is $f(x_4) = 3^2 = 9$

$$A \approx \sum_{i=1}^{4} f(x_i)\Delta x = 2.25\left(\frac{1}{2}\right) + 4\left(\frac{1}{2}\right) + 6.25\left(\frac{1}{2}\right) + 9\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)(2.25 + 4 + 6.25 + 9) = \frac{1}{2}(21.5)$$
$$= 10.75$$

Left-hand rule: first rectangle height is $f(x_0) = 1^2 = 1$
Second rectangle height is $f(x_1) = (1.5)^2 = 2.25$
Third rectangle height is $f(x_2) = (2)^2 = 4$
Fourth rectangle height is $f(x_3) = (2.5)^2 = 6.25$

$$A \approx \sum_{i=1}^{4} f(x_{i-1})\Delta x = 1\left(\frac{1}{2}\right) + 2.25\left(\frac{1}{2}\right) + 4\left(\frac{1}{2}\right) + 6.25\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)(1 + 2.25 + 4 + 6.25) = \frac{1}{2}(13.5)$$
$$= 6.75$$

Midpoint rule:

Height of the first rectangle $f(1.25) = (1.25)^2 =$
Height of second rectangle $f(1.75) = 1.75^2 = 3.0625$
Height of the third rectangle $f(2.25) = 2.25^2 = 5.0625$
Height of the fourth rectangle $f(2.75) = 2.75^2 = 7.5625$

$$A \approx \sum_{i=1}^{4} f(x_{i-1})\Delta x = 1.5625\left(\frac{1}{2}\right) + 3.0625\left(\frac{1}{2}\right) + 5.0625\left(\frac{1}{2}\right) + 7.5625\left(\frac{1}{2}\right)$$
$$= \left(\frac{1}{2}\right)(1.5625 + 3.0625 + 5.0625 + 7.5625) = \frac{1}{2}(17.25) = 8.625$$

Exact value: $\frac{26}{3} \approx 8.66666666$ ….

1. Find Delta x
2. Find the points to be evaluated (partition) $x_i = a + i\Delta x$
3. Plug in the required n points to find the height on each interval $f(x_i)$
4. Add up the heights
5. Multiply the sum by delta-x

In the limit case: step 3, will give you a formula in *i*. then apply the summation formulas to. Last step will be to take the limit.

Continue with the limit case next class.