7/27/2023

Area between 2 curves
Volumes of Revolution: Disk/Washer, Shells

In chapter 5, we learned about the relationship between the definite integral $\int_a^b f(x)dx$ and the area under the curve $f(x)$ on the interval [a,b]. (bounded by the function and the x-axis)

We are going to expand on this idea, and now find the area between any two curves, not just one curve and the x-axis.

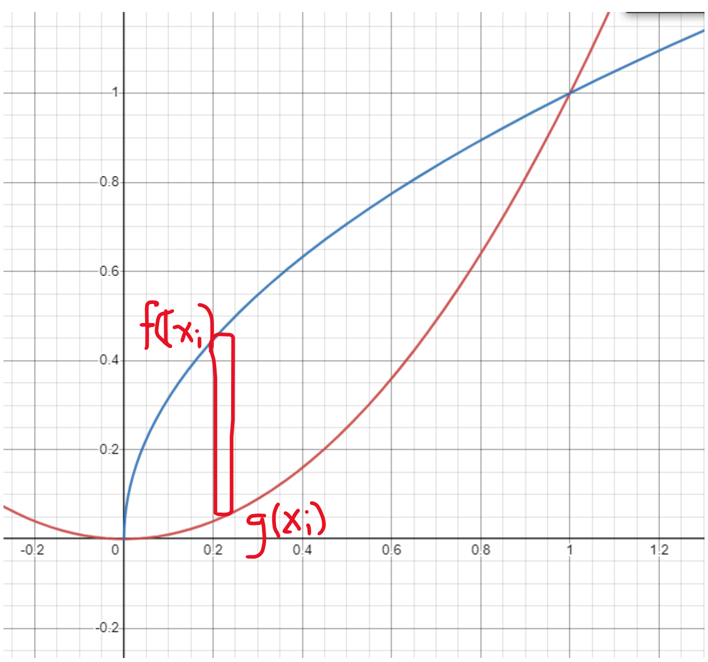
Way 1 to think about this: in some ways we've been doing this already just with a specific second curve, y=0.

$$\int_a^b f(x)dx = \int_a^b (f(x) - 0)dx$$

More generally, instead of the area bounded between f(x) and y=0, we instead replace y=0 with a second curve.

$$\int_a^b (f(x) - g(x))dx$$

This is the area between f(x) and g(x).



$f(x) = \sqrt{x}, g(x) = x^2$

When we were using y=0 as the second function, we set the f(x) equal to 0 to find the intervals (when not given), but here we set the two functions equal to each other to find the interval.

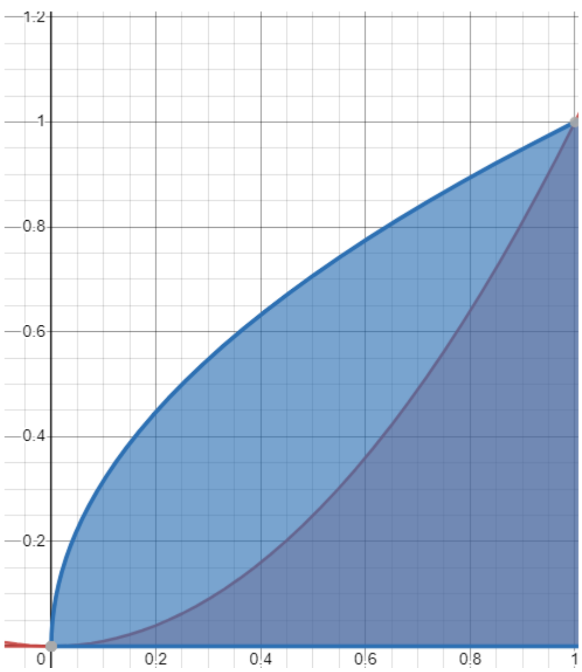
$$x^2 = \sqrt{x}$$
$$x^2 - x^{\frac{1}{2}} = 0$$
$$x^{\frac{1}{2}}\left(x^{\frac{3}{2}} - 1\right) = 0$$

$$x^{\frac{1}{2}} = 0, x = 0$$
$$x^{\frac{3}{2}} - 1 = 0, x^{\frac{3}{2}} = 1, x = 1$$

$$A = \int_0^1 \sqrt{x} - x^2 dx = \frac{2}{3}x^{\frac{3}{2}} - \frac{1}{3}x^3\Big|_0^1 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$

Way 2 is to think about the area under the top curve and then removing the part of the area under the bottom curve that we no longer want.

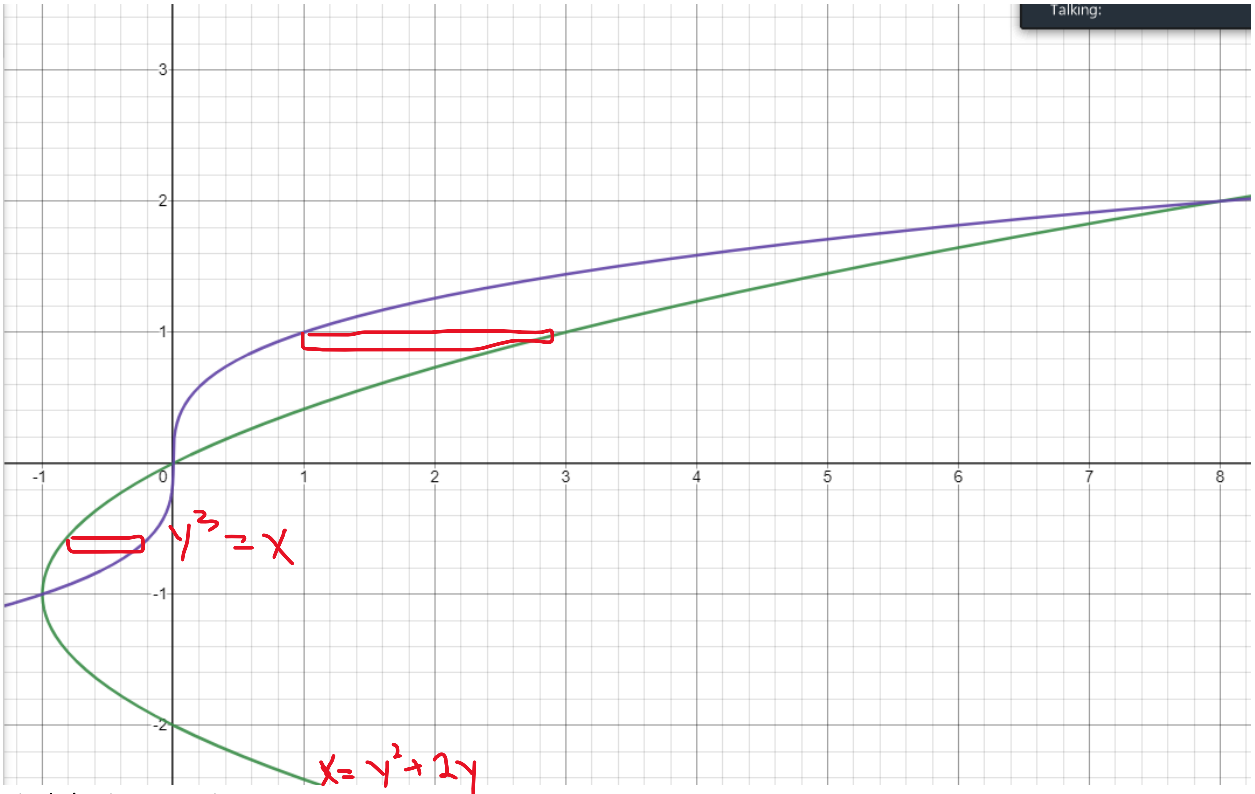
$$\int_a^b f(x)dx - \int_a^b g(x)dx = \int_a^b [f(x) - g(x)]dx$$



This produces the same definite integral we saw before and therefore also the same area.

$$\int_a^b top_f - bottom_f dx$$

Find the area between the curves, $x = y^2 + 2y, and\ x = y^3$



Find the intersections.

$$y^3 = y^2 + 2y$$
$$y^3 - y^2 - 2y = 0$$
$$y(y^2 - y - 2) = 0$$
$$y(y-2)(y+1) = 0$$

$$y = 0, -1, 2$$

When integrating in y, the top function = rightmost function, and bottom function = leftmost function
If the orientation flips, split the integral to change the order of the functions.

In the bottom region:

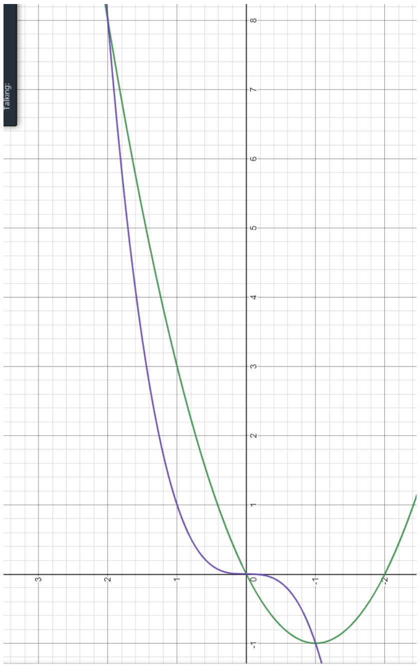
$$\int_{-1}^{0} y^3 - (y^2 + 2y)dy = \int_{-1}^{0} y^3 - y^2 - 2y\, dy = \frac{1}{4}y^4 - \frac{1}{3}y^3 - y^2\Big|_{-1}^{0} = -\left(\frac{1}{4}(1) - \frac{1}{3}(-1) - (1)\right)$$
$$= \frac{13}{12}$$

In the top region:

$$\int_{0}^{2} y^2 + 2y - y^3 dx = \frac{1}{3}y^3 + y^2 - \frac{1}{4}y^4\Big|_{0}^{2} = \frac{1}{3}(8) + 4 - \frac{1}{4}(16) = \frac{8}{3}$$

Total area= $\frac{13}{12} + \frac{8}{3} = \frac{15}{4}$

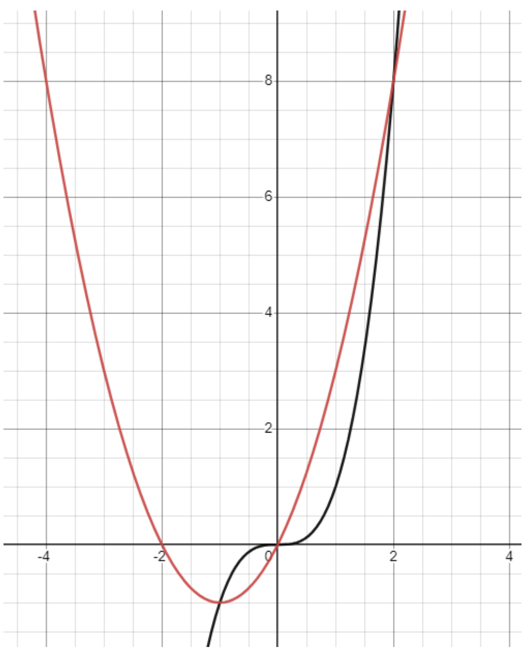
Another way to think about horizontally-oriented problems:



rotating the graph can help orient yourself with which function is "on top" and which one is "on bottom". The main drawback to this is that while the vertical axis is positive going up, the y-axis on the horizontal is positive to the left.

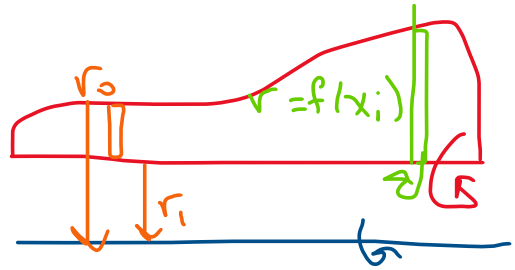
Another related alternative is to switch all the variables in all of your equations.

Instead of using $\{x = y^2 + 2y, x = y^3\}$, use $\{y = x^2 + 2x, y = x^3\}$. If you are given any other boundary lines like x=0 or y=0, you also have to flip the variables in these equations.



If set up correctly (and all variables are switched properly) these regions will have the same area as the original.

Volumes of solids of revolution



Slicing vs. shells

Slicing: disk/washer
Shells: cylindrical shells (surface area slices)

Washer and Disk Methods

If I take a rectangle bounded by the functions that define my cross section, and then I rotate around the axis (x-axis), that rectangle will become a cylinder.
The radius of the cylinder is the height of the function (relative to the axis of rotation).
The area of the face of the cylinder is the area of the circle with the given radius $A = \pi r^2$
The volume of the cylinder is the area of the face times the height (depth) is Δx or dx.

$$Volume \approx \sum_{i=1}^{n} \pi[f(x_i)]^2 \Delta x$$

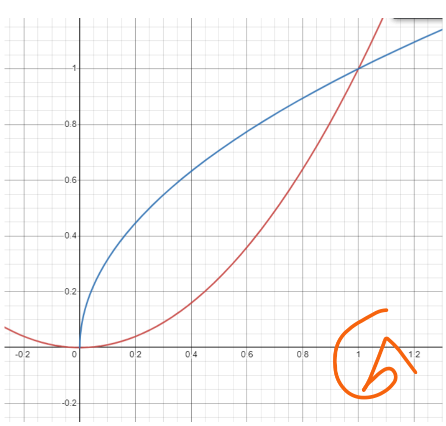
$$V = \lim_{n \to \infty} \sum_{i=1}^{n} \pi[f(x_i)]^2 \Delta x = \int_a^b \pi[f(x)]^2 dx = \pi \int_a^b [f(x)]^2 dx$$

Disk Method
This method assumes that the area being rotated touches the axis of rotation everywhere inside the interval (no gaps).

Washer Method
This method allows that there may be a gap between the region being rotated and the axis of rotation.

$$V = \pi \int_a^b \{[f(x)]^2 - [g(x)]^2\} dx$$

Find the volume of revolution from rotating the region bounded by $y = \sqrt{x}$ and $y = x^2$ around the x-axis.

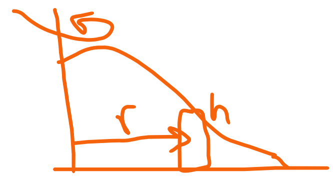


$$V = \pi \int_0^1 \left(\sqrt{x}\right)^2 - (x^2)^2 dx = \pi \int_0^1 x - x^4 dx = \pi \left[\frac{1}{2}x^2 - \frac{1}{5}x^5\right]_0^1 = \pi\left(\frac{1}{2} - \frac{1}{5}\right) = \frac{3\pi}{10}$$

When to use the disk/washer method?
When your function is defined in terms of a variable (x) and you are rotating it around the axis with the same variable (x-axis).

If we want to use this method to rotate around the y-axis, then our functions need to be defined in terms of y (solve for x).

The shell method will have the reverse relationship: function of x to rotate around the y-axis (or vice versa).



Shell Method

The foundation is not cylindrical disks, but cylindrical shells.

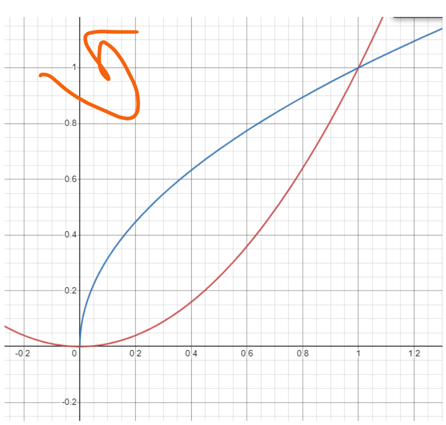
Rotating around the y-axis, we get a thin tube. When we flatten it out, it is approximately rectangle. The height of the rectangle is the height of function. The width of the rectangle is the circumference of the tube opening (related to the radius): $C = 2\pi r$. The area then is $A = 2\pi r f(x)$.
To get the volume we multiply by the thickness of the rectangle we rotated around the center, which is Δx.

$$Volume \approx \sum_{i=1}^{n} 2\pi x f(x)\Delta x$$

$$Volume = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi x f(x)\Delta x = 2\pi \int_a^b x f(x) dx$$

If the height of the cylindrical shell is the region between two curves then

$$Volume = 2\pi \int_a^b x[f(x) - g(x)]dx$$



Find the volume of the solid of revolution obtained by rotating the region bounded by $y = \sqrt{x}$ and $y = x^2$ around the y-axis.

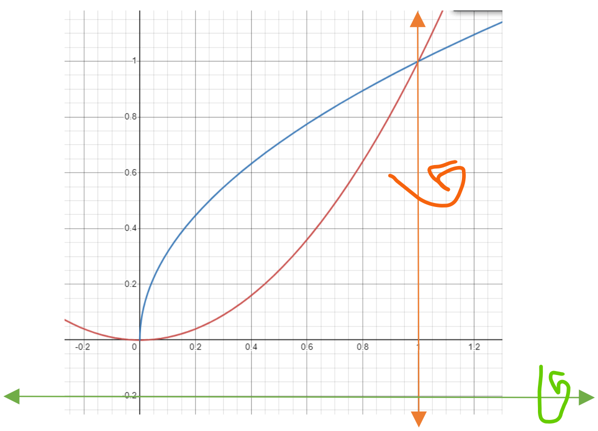
$$Volume = 2\pi \int_0^1 x\left(\sqrt{x} - x^2\right)dx = 2\pi \int_0^1 x^{\frac{3}{2}} - x^3 dx = 2\pi \left[\frac{2}{5}x^{\frac{5}{2}} - \frac{1}{4}x^4\right]_0^1 = 2\pi \left[\frac{2}{5} - \frac{1}{4}\right] = \frac{3\pi}{10}$$

(we got the same answer here because our two functions are inverses of each other… this won't normally happen)

Suppose we want to rotate our regions around a line other than the x-axis or the y-axis?

We have to make some adjustments, but the same rules will generally apply.
The lines we are rotating around will behave similarly to the axis they are parallel to …

$x = 1$ is parallel to the y-axis so you need to treat it like the y-axis. (use shell method below)
$y = -2$ is parallel to the x-axis, so you need to treat it like the x-axis. (use the washer method below)



To rotate around a line $x = c$ (outside the region you are rotating)
The radius of rotation is the only thing that changes in the shell method:

$$V = 2\pi \int_a^b (x - c)f(x)dx$$

Or

$$V = 2\pi \int_a^b (c - x)f(x)dx$$

If c is to the left of the region, use x-c, and if c is to the right of the region, use c-x.

In the washer method: what changes is radius function:
Rotating around a line $y = c$, if it is below the region:

$$V = \pi \int_a^b [f(x) - c]^2 - [g(x) - c]^2 dx$$

Or if it is above the region then

$$V = \pi \int_a^b [c - g(x)]^2 - [c - f(x)]^2 dx$$

C is the value of the axis of rotation.