7/11/2023

Continue with Riemann Sums (limit)
Definite Integrals, Properties
Fundamental Theorem of Calculus
Average Value of a function
Accumulation Functions/Derivatives

Summation formulas

$$\sum_{i=1}^{n} c = cn$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

We want to find the area under the curve of $f(x) = x^2 - x$ on the interval [1,3].

$$\int_1^3 x^2 - x \ dx$$

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

1. Delta-x: $\Delta x = \frac{b-a}{n} \rightarrow \frac{3-1}{n} = \frac{2}{n}$
2. $x_i = a + i\Delta x = a + \frac{2i}{n}$
3. $f(x_i) = \left(a + \frac{2i}{n}\right)^2 - \left(a + \frac{2i}{n}\right) = a^2 + \frac{4ai}{n} + \frac{4i^2}{n^2} - a - \frac{2i}{n} = (a^2 - a) + \left(\frac{4a-2}{n}\right) i + \left(\frac{4}{n^2}\right) i^2$
4. Area=$f(x_i)\Delta x = \left[(a^2 - a) + \left(\frac{4a-2}{n}\right) i + \left(\frac{4}{n^2}\right) i^2\right]\frac{2}{n}$
5. Summation: $\sum_{i=1}^{n} f(x_i)\Delta x =$

$$\sum_{i=1}^{n} \left[(a^2 - a) + \left(\frac{4a-2}{n}\right) i + \left(\frac{4}{n^2}\right) i^2\right]\frac{2}{n} =$$

$$\frac{2}{n}\sum_{i=1}^{n}\left[(a^2-a)+\left(\frac{4a-2}{n}\right)i+\left(\frac{4}{n^2}\right)i^2\right]=$$

$$\frac{2}{n}\left[\sum_{i=1}^{n}(a^2-a)+\sum_{i=1}^{n}\left(\frac{4a-2}{n}\right)i+\sum_{i=1}^{n}\left(\frac{4}{n^2}\right)i^2\right]=$$

$$\frac{2}{n}\left[\sum_{i=1}^{n}(a^2-a)+\left(\frac{4a-2}{n}\right)\sum_{i=1}^{n}i+\left(\frac{4}{n^2}\right)\sum_{i=1}^{n}i^2\right]=$$

$$\frac{2}{n}\left[\sum_{i=1}^{n}(1^2-1)+\left(\frac{4(1)-2}{n}\right)\sum_{i=1}^{n}i+\left(\frac{4}{n^2}\right)\sum_{i=1}^{n}i^2\right]=$$

$$\frac{2}{n}\left[\left(\frac{2}{n}\right)\sum_{i=1}^{n}i+\left(\frac{4}{n^2}\right)\sum_{i=1}^{n}i^2\right]=$$

$$\frac{2}{n}\left[\left(\frac{2}{n}\right)\frac{n(n+1)}{2}+\left(\frac{4}{n^2}\right)\frac{n(n+1)(2n+1)}{6}\right]=$$

$$\left(\frac{4}{n^2}\right)\frac{n(n+1)}{2}+\left(\frac{8}{n^3}\right)\frac{n(n+1)(2n+1)}{6}=$$

$$\frac{2n(n+1)}{n^2}+\frac{4n(n+1)(2n+1)}{3n^3}=\frac{2(n+1)}{n}+\frac{4(n+1)(2n+1)}{3n^2}=$$

$$\frac{2n}{n}+\frac{2}{n}+\frac{8n^2}{3n^2}+\frac{12n}{3n^2}+\frac{4}{3n^2}=2+\frac{2}{n}+\frac{8}{3}+\frac{4}{n}+\frac{4}{3n^2}$$

6. Take the limit as n goes to infinity

$$\lim_{n\to\infty}\sum_{i=1}^{n}f(x_i)\Delta x=\lim_{n\to\infty}\left(2+\frac{2}{n}+\frac{8}{3}+\frac{4}{n}+\frac{4}{3n^2}\right)=2+\frac{8}{3}=\frac{14}{3}$$

No longer an approximation. This is the exact value of the area.

Properties of definite integrals (as they relate to area)

If an area is matching a geometric formula, the area found by the definite integral will produce the same value as using the geometric formula.

It’s generally assumed that the area calculated is between the curve and the x-axis.

Area calculated this way comes with a sign: area that is above the x-axis is positive. But if the region we are calculating is below the x-axis, then we will get a negative sign from the definite integral. If we are asked for the geometric area, we will need to drop that sign.

If the region to be calculated is partly above the x-axis and partly below the x-axis, we will need to split up the region and calculate the areas separately so that they don't partially cancel out.

You can use properties of symmetry to simplify some of your work.

$$\int_{-a}^{a} f_{even}(x)\,dx = 2\int_{0}^{a} f_{even}(x)\,dx$$

$$\int_{-a}^{a} f_{odd}(x)\,dx = 0$$

$$\int_{a}^{b} 0\,dx = 0$$

$$\int_{a}^{b} kf(x)dx = k\int_{a}^{b} f(x)dx$$

Fundamental Theorem of Calculus

If $F(x)$ is an antiderivative of $f(x)$, then $\int_a^b f(x)dx = F(b) - F(a)$.

$$\int_{1}^{3} x^2 - x\ dx$$

$$F(x) = \frac{x^3}{3} - \frac{x^2}{2}$$

$$F(b) - F(a) = \frac{3^3}{3} - \frac{3^2}{2} - \left(\frac{1^3}{3} - \frac{1^2}{2}\right) = 9 - \frac{9}{2} - \frac{1}{3} + \frac{1}{2} = \frac{14}{3}$$

$$\int_{1}^{3} x^2 - x\ dx = \left.\frac{x^3}{3} - \frac{x^2}{2}\right|_{1}^{3} = \frac{3^3}{3} - \frac{3^2}{2} - \left(\frac{1^3}{3} - \frac{1^2}{2}\right) = 9 - \frac{9}{2} - \frac{1}{3} + \frac{1}{2} = \frac{14}{3}$$

Indefinite integral: function +C
Definite integral: number

Accumulation functions

$$F(x) = \int_{0}^{x} f(t)dt$$

$$F(x) = \int_1^x e^t \ln t \, dt$$

What if I wanted to take the derivative of an accumulation function?

$$\frac{d}{dx}[F(x)] = \frac{d}{dx}\left[\int_1^x e^t \ln t \, dt\right] =$$

“Second” Fundamental Theorem of Calculus

$$\frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x)$$

$$\frac{d}{dx}\left[\int_a^{g(x)} f(t)dt\right] = f(g(x))g'(x)$$

$$\frac{d}{dx}\left[\int_{h(x)}^{g(x)} f(t)dt\right] = f(g(x))g'(x) - f\big(h(x)\big)h'(x)$$

Relates to a property of definite integrals to split definite integrals into pieces:

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

$$\int_{h(x)}^{g(x)} f(t)dt = \int_{h(x)}^0 f(t)dt + \int_0^{g(x)} f(t)dt$$

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

$$\int_{h(x)}^{g(x)} f(t)dt = \int_0^{g(x)} f(t)dt - \int_0^{h(x)} f(t)dt$$

Example.

$$F(x) = \int_1^{x^2} e^t \ln t \, dt$$

$$\frac{d}{dx}[F(x)] = e^{x^2} \ln(x^2)\,(2x)$$

Average Value of a Function

$$\bar{f}(x) = \frac{1}{b-a}\int_a^b f(x)dx$$

Find the average value of the function $f(x) = x^2 - x$ on the interval [1,3].

$$\bar{f} = \frac{1}{3-1}\int_1^3 x^2 - x\,dx = \frac{1}{2}\left(\frac{14}{3}\right) = \frac{7}{3}$$

Find the value of the function equal to the average value.

$$f(c) = \frac{7}{3}$$

$$c^2 - c = \frac{7}{3}$$

$$c^2 - c - \frac{7}{3} = 0$$

Use the quadratic formula. And then the value of c that is inside the original interval should exist as long as the function is continuous on that interval.