6/8/2023

Derivatives of Trigonometric Functions
Chain Rule
Inverse functions??

What is the derivative of $\sin(x)$?

$$\lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

Apply the sum formula:

$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$

Rewrite the first term of the numerator with this formula.

$$\lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h} = \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin(x)}{h} =$$

$$\lim_{h \to 0} \frac{\sin x \cos h - \sin(x)}{h} + \lim_{h \to 0} \frac{\cos x \sin h}{h} =$$

$$(\sin x) \lim_{h \to 0} \frac{\cos h - 1}{h} + \cos(x) \lim_{h \to 0} \frac{\sin h}{h}$$

Special limits:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

$$\lim_{x \to 0} \frac{\cos x - 1}{x} = 0$$

$$\lim_{x \to 0} (1+x)^{\frac{1}{x}} = e \, , \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e$$

$$(\sin x) \lim_{h \to 0} \frac{\cos h - 1}{h} + \cos(x) \lim_{h \to 0} \frac{\sin h}{h} = (\sin x)[0] + \cos(x)[1] = \cos x$$

This agrees with our conjecture based on the slopes of the tangent lines to the graph.

$$\frac{d}{dx}[\sin x] = \cos x$$

$$\frac{d}{dx}[\cos x] = -\sin x$$

What the derivative of the tan (x)?

$$\frac{d}{dx}[\tan x] = \frac{d}{dx}\left[\frac{\sin x}{\cos x}\right] =$$

$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$

$$\frac{d}{dx}\left[\frac{f}{g}\right] = \frac{f'g - g'f}{g^2}$$

$$= \frac{\cos(x)\cos(x) - (-\sin(x))\sin(x)}{[\cos(x)]^2} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

$$\frac{d}{dx}[\tan x] = \sec^2 x$$

$$\frac{d}{dx}[\cot x] = -\csc^2 x$$

$$\frac{d}{dx}[\sec x] = \sec x \tan x$$

$$\frac{d}{dx}[\csc x] = -\csc x \cot x$$

The higher derivatives for sine and cosine make a pattern.

$$\frac{d}{dx}[\sin x] = \cos x$$

$$\frac{d}{dx}[\cos x] = \frac{d^2}{dx^2}[\sin x] = -\sin x$$

$$\frac{d}{dx}[-\sin x] = \frac{d^3}{dx^3}[\sin x] = -\cos x$$

$$\frac{d}{dx}[-\cos x] = \frac{d^4}{dx^4}[\sin x] = \sin x$$

We would repeat this list of values if we continue.
This is related to the fact that sine and cosine can be defined in terms of complex exponentials.

$$\cos x = \frac{\left(e^{ix} + e^{-ix}\right)}{2}$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

Recall that $i^2 = -1, i^3 = -i, i^4 = 1, i^5 = i$

This means that we can find higher order derivatives without going through the complete chain.

$$\frac{d^{74}}{dx^{74}}[\sin x] = \frac{d^2}{dx^2}\left[\frac{d^{72}}{dx^{72}}(\sin x)\right] = \frac{d^2}{dx^2}(\sin x) = -\sin x$$

Chain Rule

This rule allows us to find the derivative of composite functions.

$$h(x) = f\big(g(x)\big) = (f \circ g)(x)$$

The chain rule: $h'(x) = f'\big(g(x)\big)g'(x)$

$$h(stuff) = f(stuff)$$
$$h'(stuff) = f'(stuff)[stuff]'$$

Example.

$$h(x) = (3x^2 - 2x + 1)^5$$

$$f(x) = x^5, g(x) = \text{stuff} = 3x^2 - 2x + 1$$

$$h'(x) = 5(3x^2 - 2x + 1)^4 \times (6x - 2)$$

Example.

$$h(x) = \sqrt[3]{19x - 45}$$

$$f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}, g(x) = stuff = 19x - 45$$

$$h'(x) = \frac{1}{3}(19x - 45)^{-\frac{2}{3}} \times (19)$$

Example.

$$h(x) = \sin(x^2 + 1)$$

$$f(x) = \sin x\,, g(x) = x^2 + 1$$

$$h'(x) = \cos(x^2 + 1) \times 2x$$

Example.

$$h(x) = \cos^2(x)$$
$$h(x) = [\cos x]^2$$

$$f(x) = x^2, g(x) = \cos x$$

$$h'(x) = 2\cos x\,(-\sin x) = -2\cos x \sin x$$

Example.
Find the second derivative of $\tan(x)$

$$\frac{d}{dx}[\tan x] = \sec^2 x$$

$$\frac{d}{dx}[\sec^2 x] =$$

$$2\sec x\,(\sec x \tan x) = 2\sec^2 x \tan x$$

Compositions of more than two functions.

$$m(x) = f\left(g\big(h(x)\big)\right)$$

$$m(x) = \sin^3(6x + 11)$$

$$f(x) = x^3, g(x) = \sin(x)\,, h(x) = 6x + 11$$

$$m'(x) = 3(\sin(6x + 11))^2(\sin(6x + 11))'$$
$$= 3\sin^2(6x + 11)\cos(6x + 11)\,(6)$$

Example.

$$f(x) = \cos(\sin^4(x^2 - 1))$$

$$f'(x) = -\sin(\sin^4(x^2 - 1))\,(\sin^4(x^2 - 1))' =$$

$$-\sin(\sin^4(x^2 - 1))\,4(\sin(x^2 - 1))^3(\sin(x^2 - 1))' =$$

$$-\sin(\sin^4(x^2 - 1))\,4\sin^3(x^2 - 1)\cos\,(x^2 - 1)(2x)$$

Combinations of chain rule with product and quotient rules.

Example.

$$h(x) = x^5 \sin(x^2 + x)$$

$f(x) = x^5$	$g(x) = \sin\,(x^2 + x)$
$f'(x) = 5x^4$	$g'(x) = \cos\,(x^2 + x)(2x + 1)$

$$h'(x) = 5x^4 \sin(x^2 + x) + x^5(2x + 1)\cos(x^2 + x)$$

Doing the quotient as a product rule with a chain rule: alternative quotient rule.

Quotient rule

$$\frac{d}{dx}\left[\frac{f}{g}\right] = \frac{f'g - g'f}{g^2}$$

$$\frac{d}{dx}[fg^{-1}] = f'g^{-1} + [g^{-1}]'f = f'g^{-1} + (-1)g^{-2}g'f$$

$$= f'g^{-1} - g^{-2}g'f = \frac{f'}{g} - \frac{g'f}{g^2} = \frac{f'}{g}\left(\frac{g}{g}\right) - \frac{g'f}{g^2} = \frac{f'g - g'f}{g^2}$$

Example:

$$h(x) = \frac{x^3 - 6}{x^2 + 1}$$

$$h(x) = (x^3 - 6)(x^2 + 1)^{-1}$$

$x^3 - 6$	$(x^2 + 1)^{-1}$
$3x^2$	$-1(x^2 + 1)^{-2}(2x) = -\frac{2x}{(x^2 + 1)^2}$

$$h'(x) = 3x^2(x^2 + 1)^{-1} - \frac{2x}{(x^2 + 1)^2}(x^3 - 6)$$

Recall inverse trig functions:

$$f(x) = \sin(x)$$
$$f^{-1}(x) = \sin^{-1}(x)$$

The “power” -1 in trig functions ALWAYS means the inverse, and never the reciprocal.

$$\csc(x) = (\sin x)^{-1} = \frac{1}{\sin x}$$

$\sin^{-1} x$ is NEVER equal to $\csc x$.

Inverse functions (for trig) you put in a sine value and it outputs an angle value.

$(\sin 2x = \sin\,(2x), \sin(2x + 1) \neq \sin 2x + 1)$

$$\arcsin x = \sin^{-1} x = \operatorname{asin} x$$

$$\frac{d}{dx}[\arcsin x] = \frac{1}{\sqrt{1 - x^2}}$$

$$\frac{d}{dx}[\arccos x] = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}[\arctan x] = \frac{1}{1+x^2}$$

$$\frac{d}{dx}[\operatorname{arccot} x] = -\frac{1}{1+x^2}$$

$$\frac{d}{dx}[\operatorname{arcsec} x] = \frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}[\operatorname{arccsc} x] = -\frac{1}{x\sqrt{x^2-1}}$$

Example.

$$\frac{d}{dx}[\arcsin(4x)] = \frac{1}{\sqrt{1-(4x)^2}}(4x)' = \frac{4}{\sqrt{1-16x^2}}$$

$$\frac{d}{dx}[\operatorname{arcsec} 4x] = \frac{1}{4x\sqrt{16x^2-1}}(4) = \frac{1}{x\sqrt{16x^2-1}}$$