6/27/2023

Maxima & Minima
Graphing with Derivatives
Limits at Infinity/Asymptotes

Maxima & Minima
Singular: maximum, plural: maxima
Singular: minimum, plural: minima
Singular: extremum, plural: extrema

Local maxima, local minima, local extrema vs. absolute maxima, absolute minima, absolute extrema

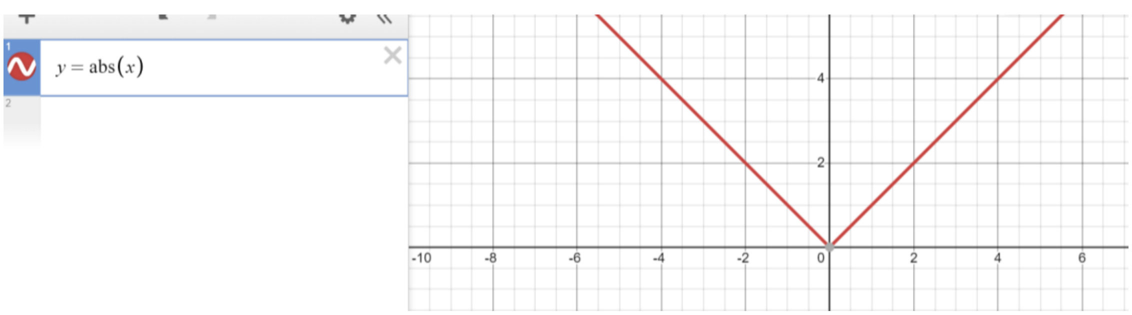
Local extrema: in the vicinity of the particular value, the extrema is either larger than all surrounding values (maximum) or it's smaller than all surrounding values (minimum).

Absolute extrema: either the whole function, or a closed interval (any time you have a closed interval for a continuous function, you must have a largest value for the function and a smallest value for function).

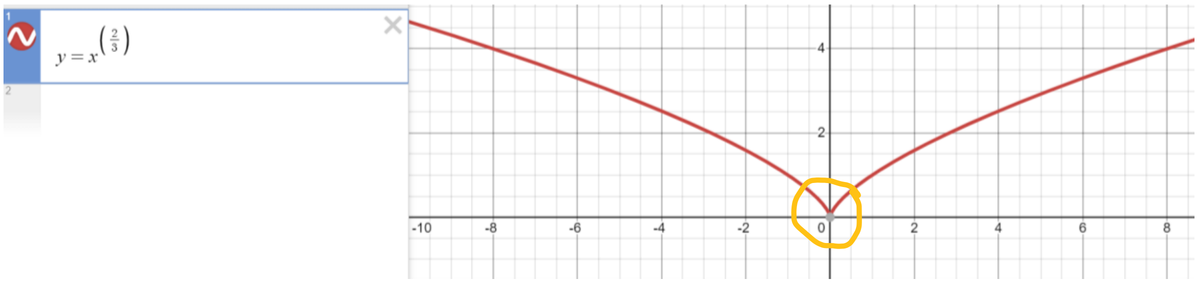
Local extrema occur at critical points:
1) The graph has a derivative = 0 at that point
2) The graphs has a undefined derivative (and is continuous) at that point (cusps).

It is possible to have a critical point that is not associated with an extrema. If the original function is not defined at some point, neither will its derivative be defined at that point, and so it can't have an extrema there either.

Steps to finding local extrema:
1) Take the derivative of the function
2) Set the derivative equal to zero
3) Solve for x
4) Check for any values where the derivative is undefined

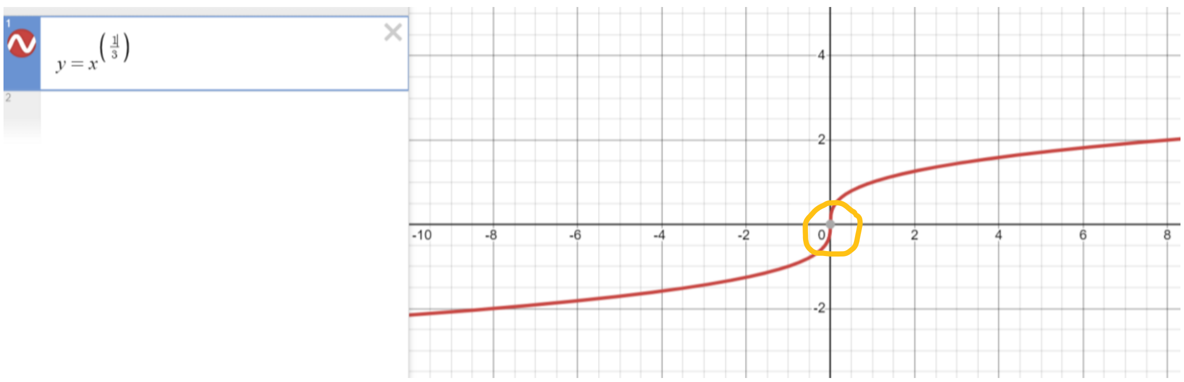


Recall that $f(x) = |x| = \begin{cases} -x, x < 0 \\ x, \ \ x \geq 0 \end{cases}$, so $f'(x) = \begin{cases} -1, x < 0 \\ 1, x > 0 \end{cases}$
The critical point is x=0 (the jump point in the derivative which is not defined).
This is a minimum



$$f(x) = x^{2/3}, f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$$

Not equal to zero anywhere, but the derivative is not defined when x=0. This is a critical point, corresponding to a minimum.

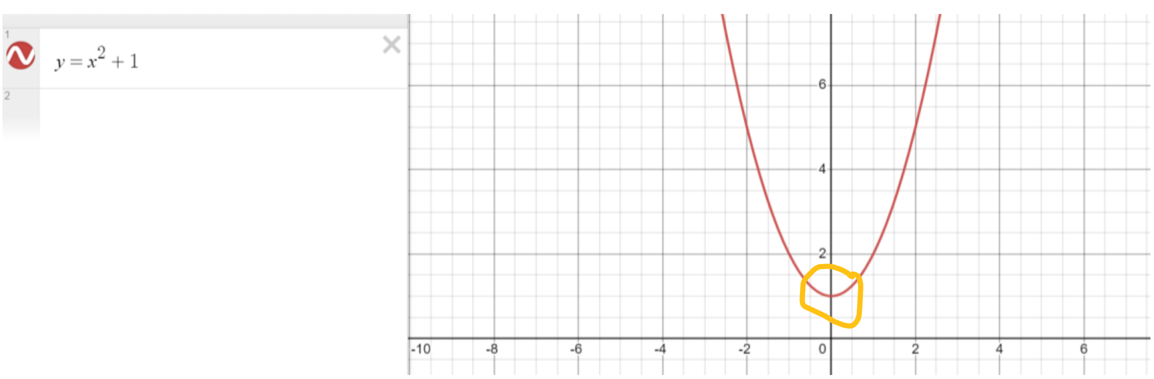


$$f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}, f'(x) = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3\sqrt[3]{x^2}}$$

Not defined when x=0, and the derivative itself can never equal zero. But the critical point doesn't correspond to a maximum or a minimum.

The critical points are going to give us candidate points, but we'll need additional information to determine whether it is min, a max or neither.

$$f(x) = x^2 + 1$$

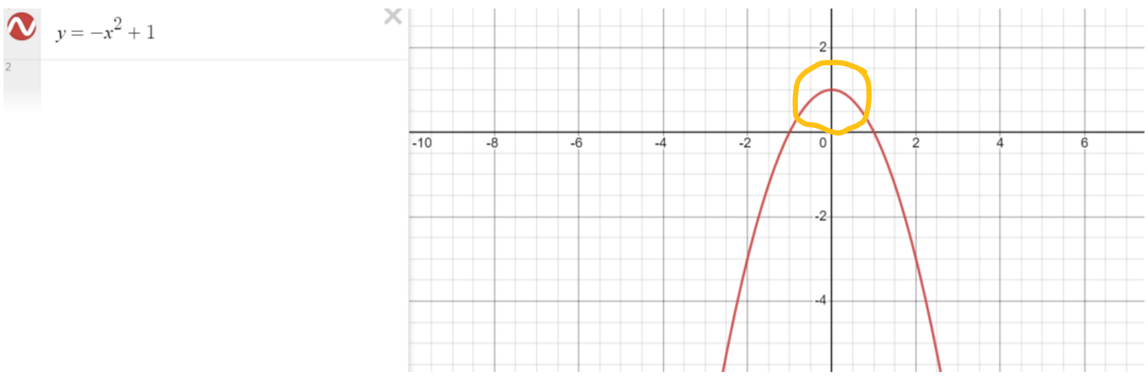


$f'(x) = 2x, 2x = 0$ implies that x=0 is the critical point. Corresponds to a minimum.

First Derivative test:

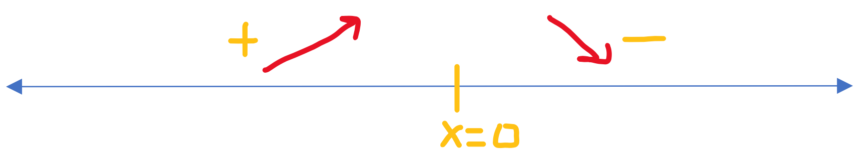
If the derivative of the function to the left of the critical is negative (the function is decreasing), and the derivative of the function to the right of the critical point is positive (the function is increasing), then this corresponds to a minimum.

If the derivative of the function to the left of the critical point is positive (the function is increasing), and the derivative of the function to right of the critical point is negative (the function is decreasing), then this corresponds to a maximum.



$f(x) = 1 - x^2, f'(x) = -2x$, -2x=0, the critical point is still x=0.

For instance, we can create a sign chart of the derivative.

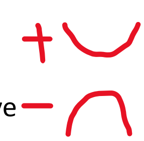


If there is no sign change, then the critical point is neither a maximum nor a minimum.

$f(x) = x^3, f'(x) = 3x^2, 3x^2 = 0$ critical point at x=0. But this isn't an extreme value because the function is increasing (the derivative is positive) on both sides.

The first derivative gave us information about whether the function was increasing or decreasing. When the derivative is positive, the function is increasing. When the derivative is negative, the function is decreasing.

The second derivative is going to give us information about concavity. A positive second derivative means that the function is concave up (it's a bowl shape opening upward). A negative second derivative means that the function is concave down (it's a bowl shape but opening downward).

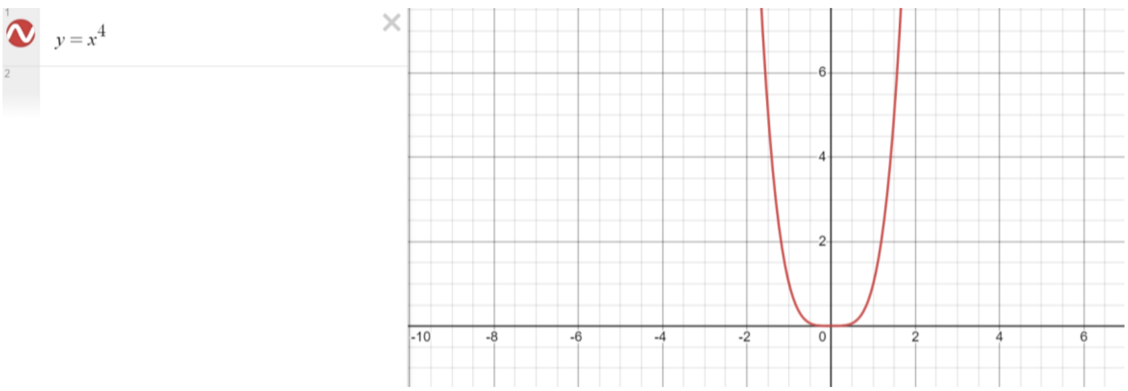


When the second derivative is equal to zero (or undefined), this is called an inflection point. This is the place where the concavity switches (can switch).

If the critical point falls on a section of the number line where the second derivative is positive (graph is concave up), then the extrema at that critical point will be a minimum.

If the critical point falls on a section of the number line where the second derivative is negative (graph is concave down), then extrema at that critical point will be a maximum.

If the critical is the same as the inflection point, then it's not determinable from this test (go back to the first derivative test).



$$f(x) = x^4, f'(x) = 4x^3, f''(x) = 12x^2$$

Critical point is x=0, and the inflection point is also x=0.

If the second derivative test fails, the first derivative test will be able to say whether it's an extremum or neither.

Graphing functions using properties of the derivatives (locating extrema, critical points, inflection points, concavity, etc.) can also combine these with algebraic elements like intercepts, asymptotes, etc.

Example.

$$f(x) = x^3 - \frac{1}{2}x^2 - 2x + 1$$

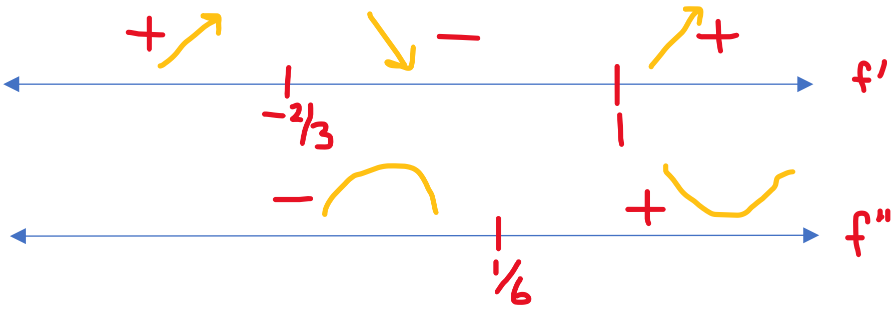
Find the critical points. Find the inflection points. Identify where the graph is concave up/down, increasing/decreasing. Graph the function.

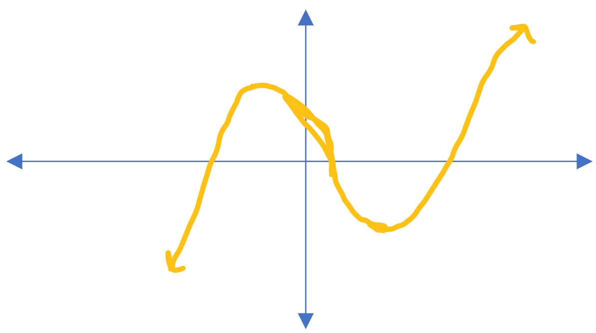
$$f'(x) = 3x^2 - x - 2$$
$$3x^2 - x - 2 = 0$$
$$(3x + 2)(x - 1) = 0$$
$$x = -\frac{2}{3}, x = 1$$

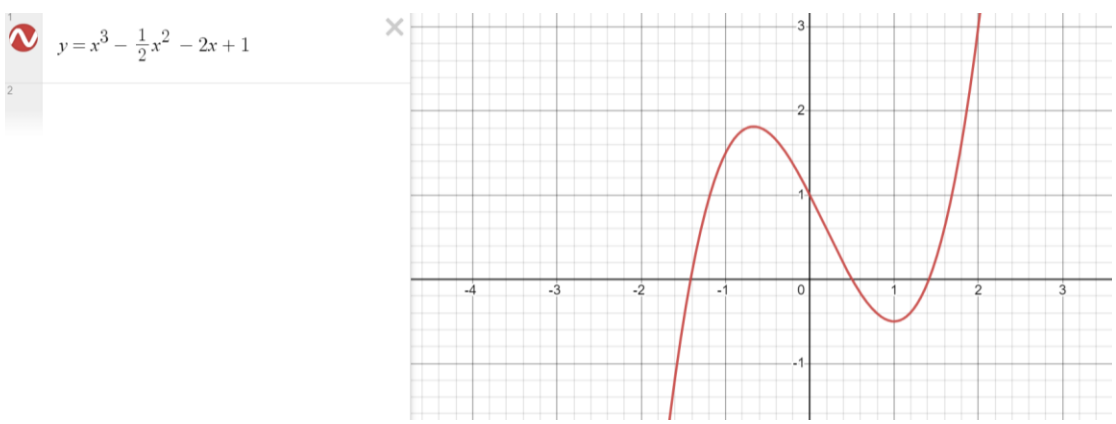
$$f''(x) = 6x - 1$$

$$x = \frac{1}{6}$$

Draw my sign charts for the two derivatives.







Example.

$$f(x) = \frac{4x}{x^2 + 1}$$

$$f'(x) = \frac{4(x^2+1) - (2x)(4x)}{(x^2+1)^2} = \frac{4x^2 + 4 - 8x^2}{(x^2+1)^2} = \frac{4 - 4x^2}{(x^2+1)^2}$$

$$4 - 4x^2 = 0$$
$$4(1 - x^2) = 4(1 - x)(1 + x) = 0$$
$$x = 1, -1$$

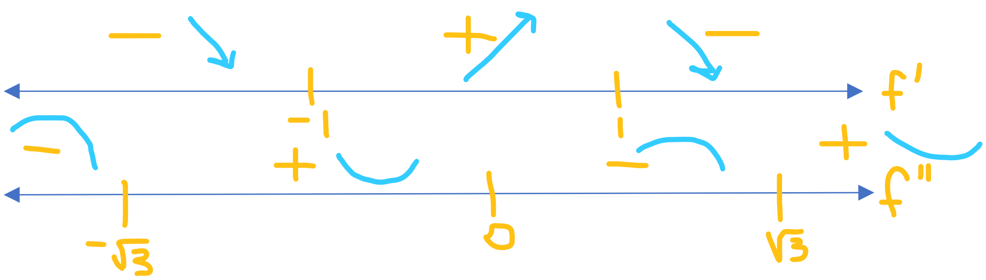
$$f''(x)=\frac{(-8x)(x^2+1)^2-2(x^2+1)(2x)(4-4x^2)}{(x^2+1)^4}=\frac{(x^2+1)\left(-8x(x^2+1)-4x(4-4x^2)\right)}{(x^2+1)^4}$$

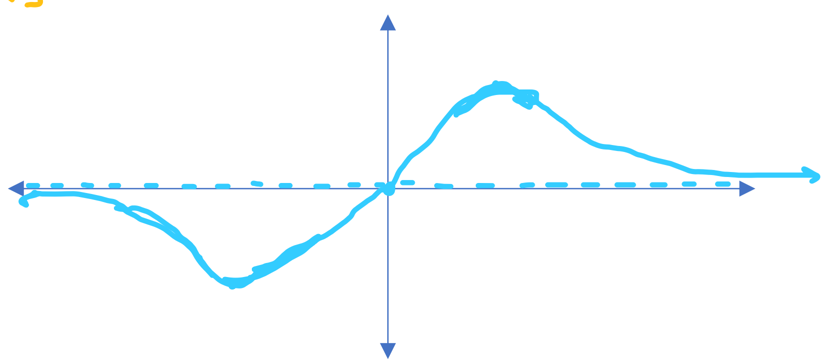
$$=\frac{-8x^3-8x-16x+16x^3}{(x^2+1)^3}=\frac{8x^3-24x}{(x^2+1)^3}=\frac{8x(x^2-3)}{(x^2+1)^3}$$

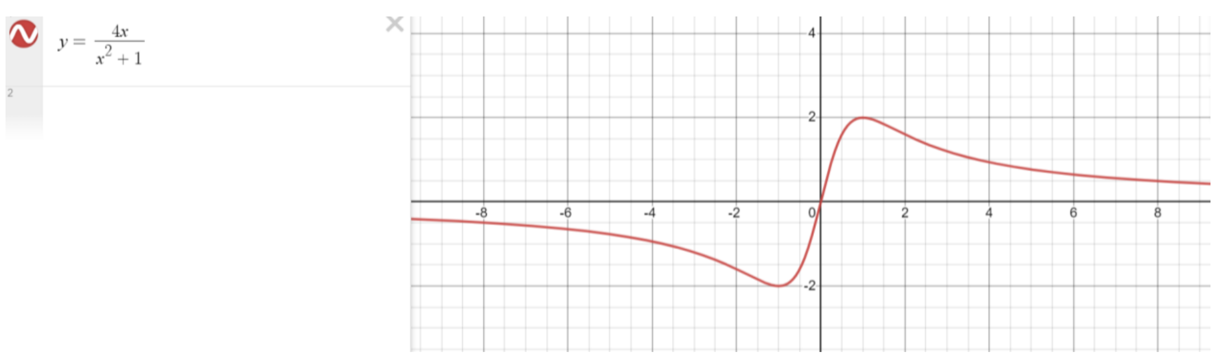
$$\frac{8x(x^2-3)}{(x^2+1)^3}=0$$

$$x=0,\pm\sqrt{3}$$

Also, there is a horizontal asymptote at y=0.







Absolute extrema.
Functions on closed intervals.
What are the absolute extrema on the function $f(x)=x^3-6x^2+5$ on the interval [0,3]?

Steps:

Check for continuity on the interval.
Take the derivative and set it equal to zero to find critical points (also check for undefined critical points).
Check the value of the function at a) any critical points in the interval, b) the endpoints of the interval.
The largest y-value is the absolute maximum. The smallest y-value is the absolute minimum. If two points give the same y-value, then both can (should) be listed as the absolute max/min.

$$f'(x) = 3x^2 - 12x = 0$$
$$3x(x-4) = 0$$
$$x = 0, x = 4$$

$x = 4$ is not on the interval, so I can ignore it.
$x = 0$ is an endpoint, so it doesn't add any additional points to test.

$f(0) = 5$ → absolute maximum
$f(3) = -22$ → absolute minimum

What are the absolute extrema on the function $f(x) = x^3 - 6x^2 + 5$ on the interval [-1,5]?

Test points: $-1, 0, 4, 5$

$$f(-1) = -2$$

$f(0) = 5$ → absolute maximum
$f(4) = -27$ → absolute minimum

$$f(5) = -20$$

Limits at infinity and asymptotes (horizontal)

For limits at infinity, the $|f(x) - L| < \varepsilon$ portion of the delta part so that there is some value of x greater than M, so that the function is within epsilon of the limit.

If a function has a limit as x goes to infinity (or negative infinity) we say that f(x) has a horizontal asymptote.

In a rational function, if the degree of the numerator and the degree of the denominator are the same, then the horizontal asymptote is the ratio of the leading coefficients. If the degree of the numerator is less than the degree of the denominator, then horizontal asymptote is y=0.

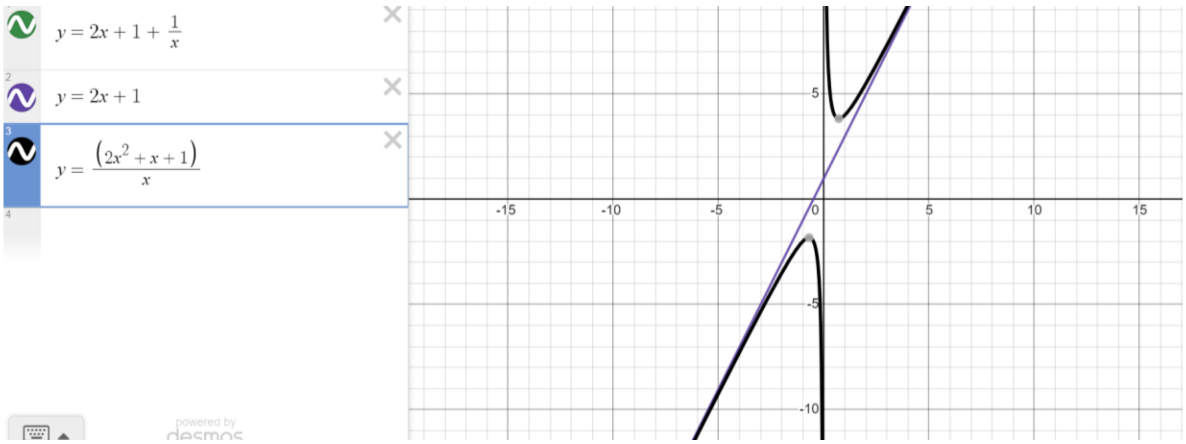
$$\lim_{x\to\infty} \frac{3x^2 - 6x + 1}{2x^2 - 7x + 11} = \lim_{x\to\infty} \frac{3x^2 - 6x + 1}{2x^2 - 7x + 11}\left(\frac{1/x^2}{1/x^2}\right) = \lim_{x\to\infty} \frac{\frac{3x^2}{x^2} - \frac{6x}{x^2} + \frac{1}{x^2}}{\frac{2x^2}{x^2} - \frac{7x}{x^2} + \frac{11}{x^2}} = \lim_{x\to\infty} \frac{3 - \cancelto{0}{\frac{6}{x}} + \cancelto{0}{\frac{1}{x^2}}}{2 - \cancelto{0}{\frac{7}{x}} + \cancelto{0}{\frac{11}{x^2}}} = \frac{3}{2}$$

There are some function types that have specific horizontal asymptotes.
The exponential function a^x has a horizontal asymptote at y=0
The arctangent function has horizontal asymptotes at $y = \pm\frac{\pi}{2}$
The hyperbolic tangent function has a horizonal asymptote at $y = \pm 1$
(logistic functions—"rational" functions that have exponentials in them also have s-shaped curves with horizontal asymptotes).

If the limit at infinity is itself infinite, then there is no upper (or lower) bound to the function. There will always be some value of x sufficiently large (or small) where the function will exceed any set value.

Limits at infinity are sometimes referred to as “end behavior”.

Oblique asymptotes/slant asymptotes: non-horizontal lines that a rational function can approach. Typically occurs when the numerator is one degree larger than the denominator. You find it by doing long division.



At the end of section 4.6 in the online textbook, they summarize all the steps (from the previous couple sections) that you can use to graph a function without a calculator. They do some good worked examples with some fairly exotic functions.