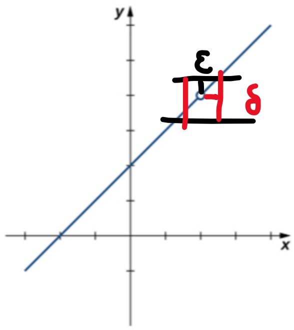
5/30/2023

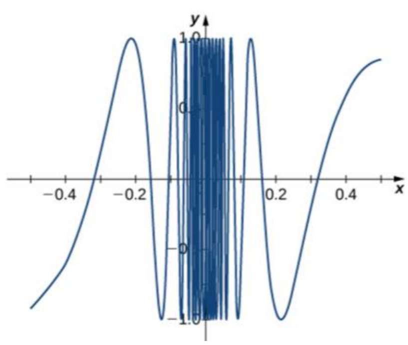
Introduction to the Course
The Limit of a Function
Limit Laws

The limit of a function is the value that the function approaches (in the y-direction) when the x-value approaches a point of interest.

We talk about the distance between x and c and a small number (δ). We talk about the distance between $f(x)$ and L (limit) also being a small value (ε). We want both epsilon and delta to be small at the same time.

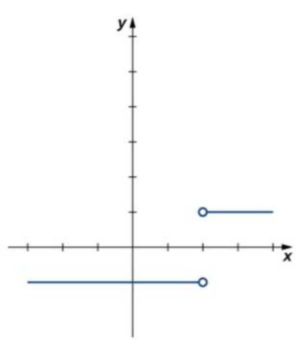


$f(x) = \frac{x^2 - 4}{x - 2}$

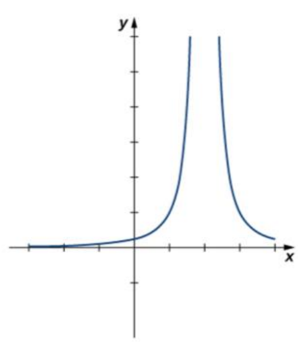


$f(x) = \sin\left(\frac{1}{x}\right)$, this graph does not have a limit at x=0.

Other examples of functions that don't have limits at a given point are functions that have a jump in them, or that have an infinite (vertical) asymptote.



$g(x) = \frac{|x - 2|}{x - 2}$



$h(x) = \frac{1}{(x - 2)^2}$

When a limit doesn't exist (when the limit from the left side (smaller values) and the limit from the right side (bigger values) don't go the same number (or the same infinity), then we label it DNE (does not exist).

"The limit as x goes to c of f(x) is equal to L" is written as:

$$\lim_{x \to c} f(x) = L$$

The limit as x goes to 1 of f(x)=3x+4 is equal to 7:

$$\lim_{x \to 1} (3x + 4) = 7$$

When the function is continuous, then the limit at a point x=c is equal to the value of the function f(c).

The left-hand limit (the limit the function is approaching through values smaller than c)

$$\lim_{x \to c^-} f(x) = L$$

$$\lim_{x \to -2^-} f(x) = L$$

The right-hand limit (the limit the function is approaching through values larger than c)

$$\lim_{x \to c^+} f(x) = L$$

Algebraic techniques we can use to simplify and find limits.
(Limit laws)

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{(x-2)(x+2)}{x - 2} = \lim_{x \to 2} \frac{(x-2)}{x-2} \lim_{x \to 2} (x + 2) = (1)(2 + 2) = 4$$

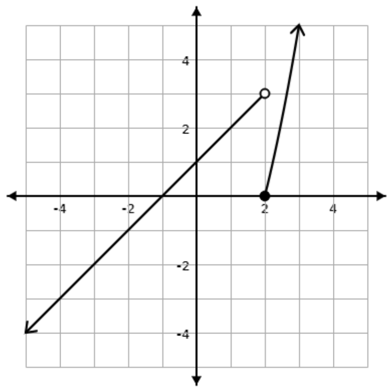
$$\lim_{x \to 9} \frac{x - 9}{\sqrt{x} - 3} = \lim_{x \to 9} \frac{(\sqrt{x} - 3)(\sqrt{x} + 3)}{\sqrt{x} - 3} = 3 + 3 = 6$$

Strategies: rationalizing or factoring can help you remove 0 denominators.
If the zero in the denominator cannot be canceled $f(x) = \frac{1}{x-2}$, then think about the graph, and where the graph goes on the left side vs. where it goes on the right side.

Piecewise defined functions and their limits.

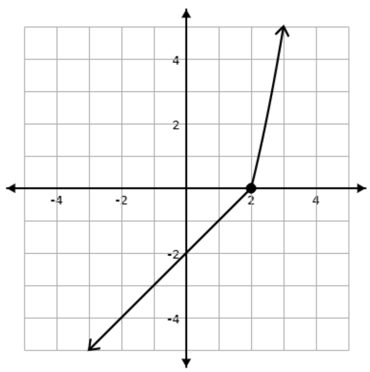
$$f(x) = \begin{cases} x + 1, x < 2 \\ x^2 - 4, x \geq 2 \end{cases}$$

https://www.graphfree.com/grapher.html



$$\lim_{x \to 2^-} f(x) = 3$$
$$\lim_{x \to 2^+} f(x) = 0$$
$$\lim_{x \to 2} f(x) = DNE$$

If I change the top function to x-2 instead of x+1



$$\lim_{x \to 2^-} f(x) = 0$$
$$\lim_{x \to 2^+} f(x) = 0$$
$$\lim_{x \to 2} f(x) = 0$$

Properties of Limits/Limit Laws

$$\lim_{x \to c}[f(x) \pm g(x)] = \lim_{x \to c} f(x) \pm \lim_{x \to c} g(x)$$

$$\lim_{x \to c} kf(x) = k \lim_{x \to c} f(x)$$

$$\lim_{x \to c} f(x)g(x) = \lim_{x \to c} f(x) \lim_{x \to c} g(x)$$

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}, \lim_{x \to c} g(x) \neq 0$$

$$\lim_{x \to c}[f(x)]^n = \left[\lim_{x \to c} f(x)\right]^n$$

$$\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to c} f(x)}$$

$$\lim_{x \to c} \ln\big(f(x)\big) = \ln\left[\lim_{x \to c} f(x)\right]$$

Example.

$$\lim_{x \to 5} \frac{\sqrt{x-1} - 2}{x - 5} \cdot \frac{\sqrt{x-1} + 2}{\sqrt{x-1} + 2} = \lim_{x \to 5} \frac{(x-1) - 4}{(x-5)\sqrt{x-1} + 2} = \lim_{x \to 5} \frac{x-5}{(x-5)\sqrt{x-1} + 2}$$

$$= \lim_{x \to 5} \frac{1}{\sqrt{x-1} + 2} = \frac{1}{4}$$

Example.

$$\lim_{x \to -3} \frac{\frac{1}{x+2} + 1}{x+3} \cdot \frac{x+2}{x+2} = \lim_{x \to -3} \frac{\frac{(x+2)}{x+2} + 1(x+2)}{(x+3)(x+2)} = \lim_{x \to -3} \frac{1 + x + 2}{(x+3)(x+2)} =$$

$$\lim_{x \to -3} \frac{x+3}{(x+3)(x+2)} = \lim_{x \to -3} \frac{1}{(x+2)} = \frac{1}{-3+2} = \frac{1}{-1} = -1$$

Issue:

$$\lim_{x \to 0} \left(\frac{1}{x} - \frac{5}{x(x-5)}\right)$$

In this case, find a common denominator.

Some functions have only one-sided limits: square roots are the main culprit.

$$\lim_{x \to 0^+} \sqrt{x}$$

Not defined for negative values of x, so the limit is only one-sided.

Squeeze Theorem

If $g(x) \leq f(x) \leq h(x)$
Then $\lim_{x \to c} g(x) \leq \lim_{x \to c} f(x) \leq \lim_{x \to c} h(x)$

$$\lim_{x \to 0} x \cos(x)$$

$$-1 \leq \cos(x) \leq 1$$

$$\lim_{x \to 0} x\,(-1) \leq \lim_{x \to 0} x \cos(x) \leq \lim_{x \to 0} x\,(1)$$

$$0 \leq \lim_{x \to 0} x \cos(x) \leq 0$$

Therefore:

$$\lim_{x \to 0} x \cos(x) = 0$$

Next time: continuity, formal definition of a limit, definition of the derivative.