MTH 263, Homework #9, Summer 2023 Name KEY

Instructions: Write your work up neatly and attach to this page. Record your final answers (only) directly on this page if they are short; if too long indicate which page of the work the answer is on and mark it clearly. Use exact values unless specifically asked to round.

1. Consider a demolition crane with a 500-pound ball suspended from a 40-foot cable that weighs 1 pound per foot. Find the work required to wind up all 40 feet of the apparatus. pg2

2. Find the centroid of the lamina of uniform density bounded by the graphs $x = 2y - y^2, x = 0$.
pg 2

3. Find the limit.

   a. $\lim_{x \to 0} \frac{\arcsin x}{x}$

   b. $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^{1/x}$

   c. $\lim_{x \to \infty} \frac{x^2}{\sqrt{x^2+1}}$

   d. $\lim_{x \to 0^+} x^3 \cot x$

   pg2

4. Determine if the integral converges or diverges. If it converges, give the value of the area under the curve.

   a. $\int_0^1 \frac{1}{\sqrt[3]{x}} dx$

   b. $\int_{-\infty}^1 e^x dx$

   c. $\int_2^\infty \frac{1}{\sqrt{x-1}} dx$

   pg 2

5. Find the centroid of the lamina of uniform density bounded by the graphs, with density $\rho = k$.

   a. $x = 2y - y^2, x = 0$

   b. $x + y = 2, x = y^2$

   c. $y = e^x, y = 0, x = 0, x = 1$

   pg 3

6. A rectangular tank with a base 4 feet by 5 feet and a height of 4 feet is full of water. The water weighs 62.4 pounds per cubic foot. How much work is done in pumping water out over the top edge in order to empty

   a. half of the tank?

   b. all of the tank?

   pg 3

7. A mining company estimates that the marginal cost of extracting x tons of copper from a mine is $C'(x) = 0.6 + 0.008x$ measured in thousands of dollars. If start-up costs are \$100,000, what is the cost of extracting the first 50 tons of copper? What about the next 50? pg 3

8. A force of 10 lbs. is required to hold a spring 4 in. beyond its natural length. How much work is done in stretching the spring from its natural length to 6 in beyond its natural length? pg 4

9. Find the average value of the function $f(x) = (x-3)^3$ on the interval [2,5]. Then find the value of c for which $f(c) = \bar{f}$.
pg 4

10. If the revenue flows into a company at a rate of $f(t) = 9000\sqrt{1+2t}$ where t is measured in years and $f(t)$ in dollars. Find the total revenue in the first 4 years.
pg 4

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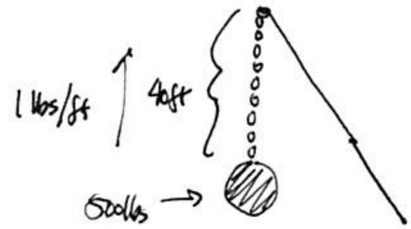
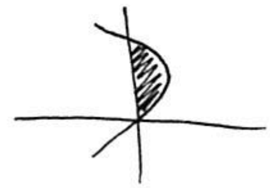
1. W on the ball $= 40 * 500 = 20{,}000$ ft·lbs



W on chain $= \int_0^{40} (40-x)1 \cdot dx = 40x - \frac{1}{2}x^2 \Big|_0^{40} = 800$ ft·lbs

total work $20{,}000 + 800 = 20{,}800$ ft·lbs

2. $M = \rho \int_0^2 2y - y^2 \, dy = \rho \left[y^2 - \frac{1}{3}y^3 \right|_0^2 = \frac{4\rho}{3}$



$M_x = \rho \int_0^2 y(2y-y^2)\,dy = \rho \int_0^2 2y^2 - y^3 \, dy =$

$\rho \left[\frac{2}{3}y^3 - \frac{1}{4}y^4 \right|_0^2 = \frac{4\rho}{3}$

$\bar{x} = \frac{M_y}{M}, \quad \bar{y} = \frac{M_x}{M}$

$M_y = \frac{\rho}{2}\int_0^2 (2y-y^2)^2\,dy = \frac{\rho}{2}\int_0^2 4y^2 - 4y^3 + y^4 \, dy = \frac{\rho}{2}\left[\frac{4}{3}y^3 - y^4 + \frac{1}{5}y^5 \right|_0^2 = \frac{\rho}{2}\left(\frac{16}{15}\right) = \frac{8\rho}{15}$

$\bar{x} = \frac{\cancel{8\rho}^{\,2}}{\cancel{15}_{\,5}} \cdot \frac{\cancel{3}}{\cancel{4\rho}} = \frac{2}{5}$ $\quad \bar{y} = \frac{4\rho}{3} \cdot \frac{3}{4\rho} = 1$

$\left(\frac{2}{5}, 1\right)$

3. a. $\lim_{x\to 0} \frac{\arcsin x}{x} = \lim_{x \to 0} \frac{\frac{1}{\sqrt{1-x^2}}}{1} = 1$

b. $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^{1/x} = L \qquad \lim_{x\to\infty} \frac{1}{x}\ln\left(1+\frac{1}{x}\right) = \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{x} = 0 = \ln L$

$\to L = 1$

c. $\lim_{x\to\infty} \frac{x^2}{\sqrt{x^2+1}} = \infty$

d. $\lim_{x\to 0^+} x^3 \cot x = \lim_{x\to 0^+} \frac{x^3}{\tan x} = \lim_{x\to 0^+} \frac{3x^2}{\sec^2 x} = \frac{0}{1} = 0$

4a. $\int_0^1 x^{-1/3}\,dx = \lim_{a\to 0} \int_a^1 x^{-1/3}\,dx \overset{\lim_{a\to 0}}{=} \frac{3}{2}x^{2/3}\Big|_a^1 = \lim_{a\to 0} \frac{3}{2}\left(1 - a^{2/3}\right) = \frac{3}{2}$ converges

b. $\int_{-\infty}^1 e^x\,dx = \lim_{a\to -\infty} \int_a^1 e^x\,dx = \lim_{a\to -\infty} e^x \Big|_a^1 = \lim_{a\to -\infty} e^1 - e^{a} = e$ converges

c. $\int_2^\infty \frac{1}{\sqrt{x-1}}\,dx = \lim_{b\to\infty} \int_2^b (x-1)^{-1/2}\,dx = \lim_{b\to\infty} 2(x-1)^{1/2}\Big|_2^b = \lim_{b\to\infty} 2\left(\sqrt{b-1} - \sqrt{2-1}\right) = \infty$
diverges

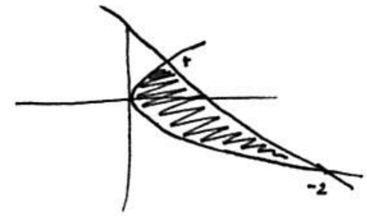
5a. this is the same as #2.

b. $x+y=2 \quad x=y^2 \qquad x=2-y$

$y^2+y-2=0$

$(y+2)(y-1)=0$

$y=-2,\ y=1$



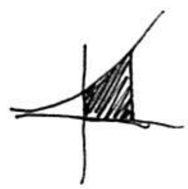
$$M=\int_{-2}^{1}(2-y)-y^2\,dy = 2y-\tfrac{1}{2}y^2-\tfrac{1}{3}y^3\Big|_{-2}^{1}=\frac{9}{2}$$

$$M_x=\int_{-2}^{1}y(2-y)-y(y^2)\,dy=\int_{-2}^{1}2y-y^2-y^3\,dy = y^2-\tfrac{1}{3}y^3-\tfrac{1}{4}y^4\Big|_{-2}^{1}=\frac{-9}{4}$$

$$M_y=\frac{1}{2}\int_{-2}^{1}(2-y)^2-(y^2)^2\,dy=\frac{1}{2}\int_{-2}^{1}4-4y+y^2-y^4\,dy=\frac{1}{2}\Big[4y-2y^2+\tfrac{1}{3}y^3-\tfrac{1}{5}y^5\Big|_{-2}^{1}=\frac{72}{5}$$

$$\bar{x}=\frac{M_y}{M}=\frac{\cancel{72}^{8}}{5}\cdot\frac{2}{\cancel{9}}=\frac{16}{5} \qquad \bar{y}=\frac{M_x}{M}=\frac{-\cancel{9}}{\cancel{4}_2}\cdot\frac{\cancel{2}^1}{\cancel{9}}=\frac{-1}{2} \qquad \left(\frac{16}{5},-\frac{1}{2}\right)$$

c. $M=\int_0^1 e^x\,dx = e^x\Big|_0^1=e-1$



$$M_x=\int_0^1 xe^x\,dx = xe^x-e^x\Big|_0^1=e-e-0+1=e+1$$

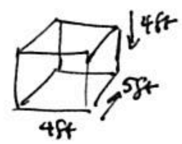
you can do this numerically

$$M_y=\frac{1}{2}\int_0^1 e^{2x}\,dx=\frac{1}{2}\cdot\frac{1}{2}e^{2x}\Big|_0^1=\frac{1}{4}e^2-\frac{1}{4}=\frac{1}{4}(e^2-1)=\frac{1}{4}\frac{(e-1)(e+1)}{1}$$

$$\bar{x}=\frac{1}{4}\frac{(e^2-1)}{e-1}=\frac{e+1}{4}=\frac{M_y}{M} \approx 0.92957 \qquad \bar{y}=\frac{M_x}{M}=\frac{e+1}{e-1}\approx 2.1639$$

6a. $\int_0^2 62.4\cdot 20\cdot(4-x)\,dx = 1248\left[4x-\tfrac{1}{2}x^2\right|_0^2 = 7488$ ft pounds

if the tank is only half full



$\int_2^4 62.4\cdot 20\cdot(4-x)\,dx = 1248\left[4x-\tfrac{1}{2}x^2\right|_2^4 = 2496$ ft pounds

if the tank is full and you are only pumping out the top half

b. $\int_0^4 62.4\cdot 20\cdot(4-x)\,dx = 1248\left[4x-\tfrac{1}{2}x^2\right|_0^4 = 9984$ ft pounds full tank

7. $\int_0^{50} 0.6+0.008x\,dx = 0.6x+.004x^2\Big|_0^{50}=40$ + fixed costs for startup

$\int_{50}^{100} 0.6+0.008x\,dx = 0.6x+.004x^2\Big|_{50}^{100}=60$

8. $F = kx$ $\quad 4\text{in} = \frac{1}{3}\text{ ft}$ $\quad 6\text{in} = \frac{1}{2}\text{ ft}$

$10 = k\left(\frac{1}{3}\right)$

$k = 30$ $\quad \int_0^{1/2} 30x\,dx = 15x^2\Big|_0^{1/2} = \frac{15}{4}$ ft.pounds

9. $\frac{1}{5-2}\int_2^5 (x-3)^3\,dx = \frac{1}{3}\cdot\frac{1}{4}(x-3)^4\Big|_2^5 = \frac{1}{\cancel{3}}\cdot\frac{\cancel{15}^{\,5}}{4} = \frac{5}{4}$

$(x-3)^3 = \frac{5}{4}$ $\quad x - 3 = \sqrt[3]{5/4} \rightarrow x = 3 + \sqrt[3]{5/4}$

10. $\int_0^4 9000\sqrt{1+2t}\,dt$ $\quad u = 1+2t$ $\quad \int \sqrt{1+2t}\,dt = \int \frac{u^{1/2}}{2}\cdot du = \frac{1}{2}\cdot\frac{2}{3}u^{3/2}$

$du = 2dt$

$\frac{1}{2}du = dt$

$\frac{1}{3}\cdot 9000\,(1+2t)^{3/2}\Big|_0^4 = 3000\left[9^{3/2} - 1\right] = 3000\left[27-1\right] = 3000\cdot 26 = \$78{,}000$