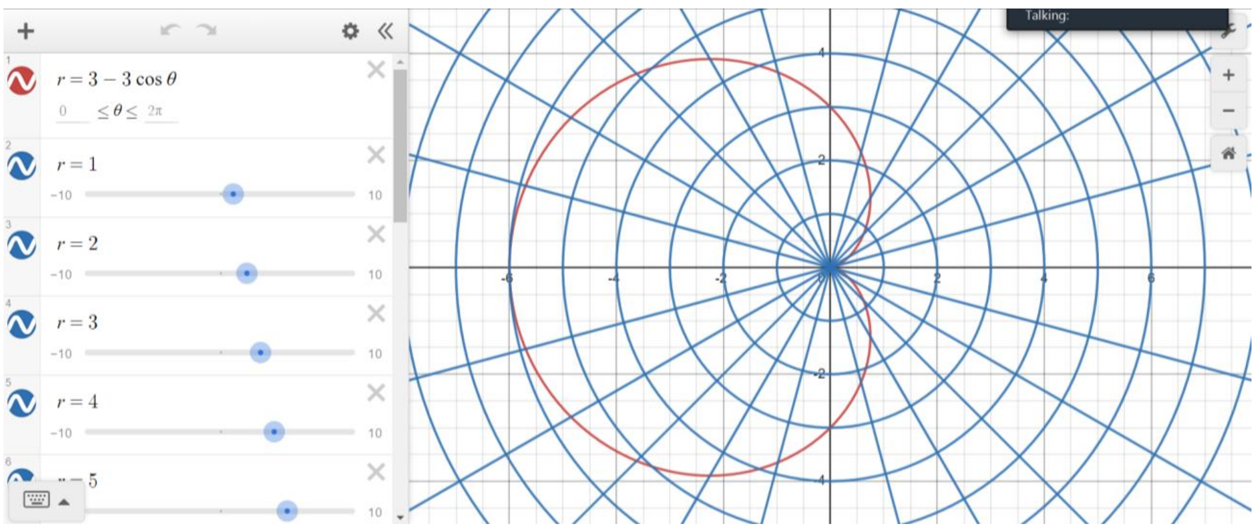
8/1/2022

Polar Coordinates
Calculus with Polar Coordinates
Conic Sections

Polar Coordinates:
A point in the plane is designated by a pair of coordinates that depend on the angle from the positive x-axis (θ) and the distance from the origin (r). Points are generally given in the form (r, θ). Theta is the independent variable when we consider functions, so r is a function of theta. Generally speaking, unless otherwise stated, the theta variable is always in radians.

Polar Graphs in Desmos
https://www.desmos.com/calculator/ms3eghkkgz?lang=zh-TW



Converting between rectangular coordinates and polar coordinates.

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$x^2 + y^2 = r^2$$
$$\theta = \tan^{-1}\frac{y}{x}$$

Converting points.
Convert the point (1,1)=(x,y) from rectangular to polar.

$$1^2 + 1^2 = r^2 = 2$$
$$r = \sqrt{2}$$

$$\theta = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$$

Point in polar is $\left(\sqrt{2}, \frac{\pi}{4}\right)$.

Polar coordinates are not unique. $\left(\sqrt{2}, \frac{9\pi}{4}\right)$ is the same point as above. Another example $\left(-\sqrt{2}, \frac{5\pi}{4}\right)$.

Angles in the 2nd and 3rd quadrants may need you to add π to the angle to get it into the correct quadrant. Inverse tangent only produces angles in the 1st and 4th quadrants.

Convert the point $\left(3, \frac{\pi}{6}\right)$ to rectangular coordinates.

$$x = 3\cos\left(\frac{\pi}{6}\right) = 3\left(\frac{\sqrt{3}}{2}\right) = \frac{3\sqrt{3}}{2}$$
$$y = 3\sin\left(\frac{\pi}{6}\right) = 3\left(\frac{1}{2}\right) = \frac{3}{2}$$

Point in rectangular coordinates is $\left(\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$.

Converting functions from rectangular to polar and from polar to rectangular.

For polar functions, try to solve for r (since that is the function variable). For rectangular functions, try to solve for y. (It may not be possible, in which case you can leave it.)

Convert the function $y = x^2 + 3$ to polar coordinates.

$$r\sin\theta = r^2\cos^2\theta + 3$$

If I had just $y = x^2$

$$r\sin\theta = r^2\cos^2\theta$$

Divide both sides by r

$$\sin\theta = r\cos^2\theta$$

$$r = \frac{\sin\theta}{\cos^2\theta} = \tan\theta\sec\theta$$

Converting from polar to rectangular.

$$r = 4\cos\theta$$

Multiply by r

$$r^2 = 4r\cos\theta$$
$$x^2 + y^2 = 4x$$

This is a circle shifted off the origin.

Plotting polar curves.

When we go to calculate the area of a petal or a loop, the loops/petals begin and end where the graph crosses the origin. So, to find the angles you need for the integration, set the polar equation equal to zero (where the radius is zero). Solve for the angles.

$$r = 4\cos 3\theta$$
$$0 = 4\cos 3\theta$$
$$0 = \cos 3\theta$$

Ask yourself, where is $\cos t = 0$?
At odd multiples of $\frac{\pi}{2}$.... $\frac{\pi}{2}, \frac{3\pi}{2}, -\frac{\pi}{2}$, ... etc.
Set these values equal to 3θ

$$3\theta = \frac{\pi}{2} \to \theta = \frac{\pi}{6}$$

So experiment with graphing different kinds of polar functions.

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

One of the weird things about this formula is that the tangent line will be in rectangular coordinates.

$$r = 4\cos 3\theta$$

Find the equation of the tangent line at $\theta = \frac{\pi}{12}$.

$$r = 4\cos\left(3 \times \frac{\pi}{12}\right) = 4\cos\left(\frac{\pi}{4}\right) = \frac{4}{\sqrt{2}} = 2\sqrt{2}$$

$$\frac{dr}{d\theta} = -12\sin 3\theta$$

$$\frac{dy}{dx} = \frac{(-12\sin 3\theta)\sin\theta + 4\cos\ 3\theta(\cos\theta)}{(-12\sin 3\theta)cos\theta - 4\cos 3\theta\ (\sin\theta)} = \frac{(-6\sqrt{2})\left(\sin\frac{\pi}{12}\right) + 2\sqrt{2}\cos\left(\frac{\pi}{12}\right)}{(-6\sqrt{2})\left(\cos\frac{\pi}{12}\right) - 2\sqrt{2}\sin\left(\frac{\pi}{12}\right)} = \frac{0.5358983849}{-8.92820323}$$
$$= -0.0600 \ldots$$

This is the slope of the tangent line at that point.
And this appears to agree with the graph.
If we wanted to find the equation of the tangent line (and not just the slope), you would need to convert the point $\left(2\sqrt{2}, \frac{\pi}{12}\right)$ back to rectangular coordinates and then use your line equation:

$$y - y_1 = m(x - x_1)$$

But usually the questions don’t go this far. They stop with the slope.

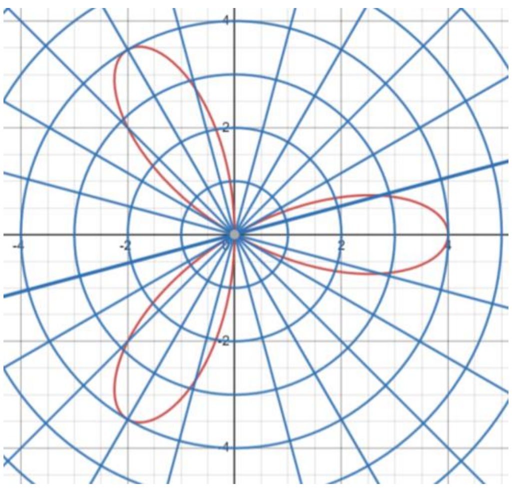
Finding the area of a section of polar graph.

$$A = \frac{1}{2}\int_{\theta_1}^{\theta_2} r^2 d\theta$$

If you are finding the area between two curves, then

$$A = \frac{1}{2}\int_{\theta_1}^{\theta_2} (r_1^2 - r_2^2) d\theta$$

Find area of one petal of the function $r = 4\cos 3\theta$.



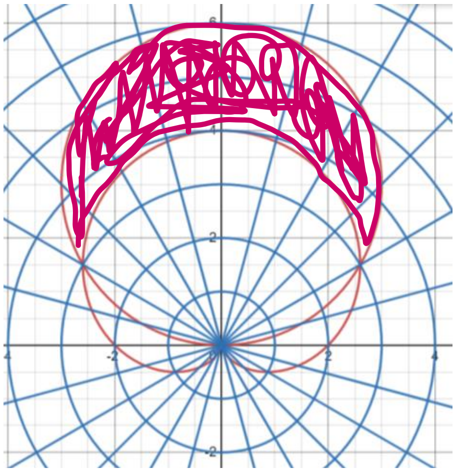
Earlier, we solved for where this graph crossed the origin (what defines the petals). $\theta = \frac{\pi}{6}, -\frac{\pi}{6}$

$$A = \frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} (4\cos 3\theta)^2 d\theta = \frac{1}{2} \times 2\int_0^{\frac{\pi}{6}} 16\cos^2 3\theta \, d\theta$$

$$16\int_0^{\frac{\pi}{6}} \frac{1}{2}(1 + \cos 6\theta) d\theta = 8\left[\theta + \frac{1}{6}\sin 6\theta\right]_0^{\frac{\pi}{6}} = 8\left(\frac{\pi}{6}\right) = \frac{4\pi}{3}$$

The area between two polar curves.

Find the area between $r = 2 + 2\sin\theta$, $r = 6\sin\theta$.


We want the area outside the cardioid and inside the circle.

Where do the curves intersect with each other?

$$2 + 2\sin\theta = 6\sin\theta$$
$$2 = 4\sin\theta$$
$$\frac{1}{2} = \sin\theta$$

$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$A = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (6\sin\theta)^2 - (2+2\sin\theta)^2 d\theta = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} 36\sin^2\theta - (4 + 8\sin\theta + 4\sin^2\theta)d\theta =$$

$$\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} 32\sin^2\theta - 4 - 8\sin\theta\, d\theta = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} 32\left(\frac{1}{2}\right)(1-\cos 2\theta) - 4 - 8\sin\theta\, d\theta$$

And so it goes…

These problems are not so hard if you are careful, but they can be tedious.

Arc length of polar graphs.

$$s = \int_{\alpha}^{\beta} \sqrt{[f(\theta)]^2 + [f'(\theta)]^2} d\theta = \int_{\alpha}^{\beta} \sqrt{r^2 + \left[\frac{dr}{d\theta}\right]^2}\, d\theta$$

Find the length of the curve $r = 6\sin\theta$ on the interval $[0, \pi]$.

$$s = \int_{\alpha}^{\beta} \sqrt{r^2 + \left[\frac{dr}{d\theta}\right]^2}\, d\theta = \int_0^{\pi} \sqrt{(6\sin\theta)^2 + [6\cos\theta]^2} d\theta = \int_0^{\pi} \sqrt{36\sin^2\theta + 36\cos^2\theta}\, d\theta$$

$$= \int_0^{\pi} \sqrt{36(\sin^2\theta + \cos^2\theta)} d\theta = \int_0^{\pi} \sqrt{36} d\theta = \int_0^{\pi} 6 d\theta = 6\pi$$

This is a circle of radius ½ the coefficient, so the radius is 3.
Circumference is the arc length = $2\pi r = 2\pi(3) = 6\pi$

Conic Sections

Circle $(x-h)^2 + (y-k)^2 = r^2$ (two squared terms, both have the same coefficient, and both are the same sign)
Ellipse $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ (two squared terms, both are the same sign, different coefficients)
Hyperbola $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ (two squared terms, but they are different signs)
Parabola $(y-k) = \frac{1}{4p}(x-h)^2$ (has only one squared term)

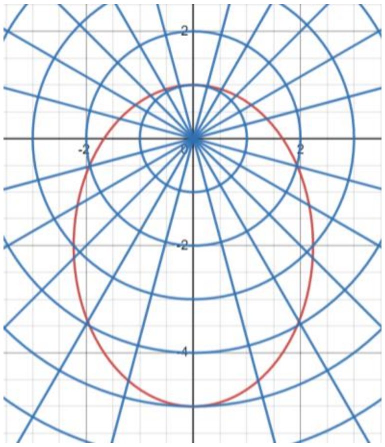
Conic sections in polar coordinates.

$$r = \frac{ep}{1 \pm e\cos\theta}, r = \frac{ep}{1 \pm e\sin\theta}$$

e is the eccentricity
The eccentricity of a circle is 0
The eccentricity of an ellipse is $0 < e < 1$
The eccentricity of a parabola is 1
The eccentricity of a hyperbola is $e > 1$

$$r = \frac{5}{3 + 2sin\theta} = \frac{\frac{5}{3}}{1 + \frac{2}{3}\sin\theta}$$

This is an ellipse because $e = \frac{2}{3}$



What is happening on Wednesday – our last lecture!! –
Vectors (for problems 6-13) on Homework.
Whatever time is left we can talk about the final.