7/6/2022

Sequences (5.1)
Introduction to Infinite Series (5.2)

Sequences a list of numbers in a particular order, they are often derived from a formula for the nth term in the sequence.

$$a_n = \frac{n}{n+1}$$

If we start with n=0

$$0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots$$

This is sometimes referred to as an explicit formula (you can jump to any term in the sequence without generating the whole sequence).

Can also be derived recursively.
These are sometimes called implicit formulas.

$$a_0 = 1, a_1 = 1, a_{n+1} = a_n + a_{n-1}$$

$$1,1,2,3,5,8,13, \dots.$$

Fibonacci sequence

Arithmetic and geometric sequence
Arithmetic sequence is a sequence where the next term in the sequence differs from the previous term by additive value which is constant.

$$3, 5, 7, 9, 11, \dots$$

The constant which we are adding is called a constant difference.
Recursive version:

$$a_0 = 3, a_{n+1} = a_n + 2$$

(generally $a_{n+1} = a_n + d$)

We can also write this as an explicit formula.

$$a_n = (d)n + a_0$$

In this case

$$a_n = 2n + 3$$

Geometric sequence
We multiply by a constant each time from term to term.

$$1,2,4,8,16,32,64, \dots$$

Recursive version is $a_0 = 1, a_{n+1} = 2a_n$

Explicit formula $a_n = a_0(r)^n$, so here that's $a_n = 2^n$

So a sequence like 3,6,12,24,48, … would have a formula of $a_n = 3(2)^n$.

Other sequences exist and don't follow these patterns,

$$a_n = n(n+1)$$

$$0,2,6,12,20,\ldots$$

Suppose we want to generate a formula for a given sequence.

$$\frac{1}{2},\frac{3}{4},\frac{7}{8},\frac{15}{16},\frac{31}{32},\ldots$$

$$a_n = \frac{2^n - 1}{2^n} = 1 - \frac{1}{2^n}$$

Factorials are common features.
Factorial is a product of all the integers from the starting point to 1.

$$7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$$

1, 2, 6, 24, 120, 720, 5040, …

$$0! = 1$$

A sequence like:

$$\frac{1}{1},\frac{3}{2},\frac{9}{6},\frac{27}{24},\frac{81}{120},\frac{243}{720},\frac{729}{5040},\ldots, a_n = \frac{3^n}{(n+1)!} \ or \frac{3^{n-1}}{n!}$$

Pattern generation skill will carry over.

We can talk about the limit of a sequence much the same way we talk about the limit of a function.

$$\lim_{n\to\infty} a_n$$

If the limit goes to a finite number, then we say the sequence converges.
If the limit goes to infinity (or negative infinity) or it goes nowhere, then we say it diverges.

$$\lim_{n\to\infty} \frac{2^n - 1}{2^n} = 1$$

$$\lim_{n\to\infty} \frac{3^n}{(n+1)!} = 0$$

$$\lim_{n\to\infty} \frac{(-1)^n n}{n+1} = diverges$$

$$0, -\frac{1}{2}, \frac{3}{4}, -\frac{4}{5}, \frac{5}{6}, -\frac{6}{7}, \ldots$$

Bounded and monotonic.

If a sequence is bounded there is a value below which it never falls, and a value above which it never rises.
For example, a sequence that is always positive is bounded below by 0.

$$\frac{1}{1}, \frac{3}{2}, \frac{9}{6}, \frac{27}{24}, \frac{81}{120}, \frac{243}{720}, \frac{729}{5040}, \ldots$$

There is also an upper bound. We can use the largest value in the sequence if one exists.
Here the upper bound is 1.5 or we could use any value larger than that.

$$\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \frac{31}{32}, \ldots$$

This one also have a lower bound of 0. And an upper bound of 1. (it is bounded above by 1: $1 - \frac{1}{2^n} = a_n$)

Monotonic. It's always increasing or always decreasing.

$$\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \frac{31}{32}, \ldots$$

This sequence is always increasing. It is monotonic.

$$\frac{1}{1}, \frac{3}{2}, \frac{9}{6}, \frac{27}{24}, \frac{81}{120}, \frac{243}{720}, \frac{729}{5040}, \ldots$$

This sequence is not monotonic.
Not if we consider the whole sequence.
The sequence is monotonic after a certain point in the sequence, here after the third term.

$$0, -\frac{1}{2}, \frac{3}{4}, -\frac{4}{5}, \frac{5}{6}, -\frac{6}{7}, \ldots$$

This sequence is never monotonic.

Theorem of sequence convergence says that if a sequence is bounded and monotonic (at least at some point it remains monotonic), then the sequence converges.

If the sequence is monotonic increasing, then we must find an upper bound. If the sequence is monotonic decreasing, then we must be able to find the lower bound.

The monotonic property can omit a finite number of terms of the sequence, but at some point must remain only increasing or only decreasing.

Infinite Series.

An infinite series is the sum of a sequence.

$$\sum_{n=1}^{\infty} a_n$$

We are going to look at a series of convergence or divergence tests.

The easiest test for infinite series is the Test of Divergence.

Given an infinite series $\sum_{n=1}^{\infty} a_n$, if the $\lim_{n\to\infty} a_n \neq 0$ then the sequence diverges.

We have to careful about the converse. If the limit is 0, this does not guarantee that the sum is finite.

What about geometric series are a special case.

$$\sum_{n=0}^{\infty} a(r^n)$$

The convergence of the series depends entirely on the value of r. If $|r| < 1$, the series will converge. If the $|r| \geq 1$, the series diverges.

Suppose $r = \frac{1}{2}$

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

Suppose $r = \frac{3}{2}$

$$\sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n = 1 + \frac{3}{2} + \frac{9}{4} + \frac{27}{8} + \cdots$$

Formula for the value that the convergent series with add to.

$$\sum_{n=0}^{\infty} a(r^n) = \frac{a}{1-r}$$

Telescoping series
We need partial fractions for this.

$$a_n = \frac{1}{(n+1)(n+2)}$$

As an example.
Factors in the denominator that are integer steps apart.

Examples.

$$a_n = \frac{1}{(2n+1)(2n+3)}$$

$$a_n = \frac{1}{(n)(n+3)}$$

$$a_n = \ln\left(\frac{n}{n+1}\right) = \ln(n) - \ln(n+1)$$

Does the telescoping series

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)}$$

Converge or diverge?

$$\frac{1}{(n+1)(n+2)} = \frac{A}{n+1} + \frac{B}{n+2}$$

$$A(n+2) + B(n+1) = 1$$
$$n = -1$$
$$A(1) = 1$$
$$n = -2$$
$$B(-1) = 1$$
$$A = 1, B = -1$$

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)} = \sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2}\right) =$$

$$\left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \cdots + \frac{1}{n+2}$$

Infinite sum is the first term + the limit of the last term.

$$\frac{1}{2} + \lim_{n\to\infty} \frac{1}{n+2} = \frac{1}{2}$$

If the factors differ by 1, then the sum is just the first term and the limit of the second term.
If the factors differ by 2, then the sum is the first two term and the limit of the last two term.
Etc.

Next time: more tests of convergence.