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Parametric Equations

A parametric set of equations defines a curve (in 2D) by relating the variables x and y to another variable called a parameter. Mostly we are going to use t as the parameter. We can think of t as the independent variable and x and y both as dependent variables.

One of the advantages is that we can treat curves that are not functions of x (y is not explicitly defined in terms of x, or is not a function of x) as function of time.

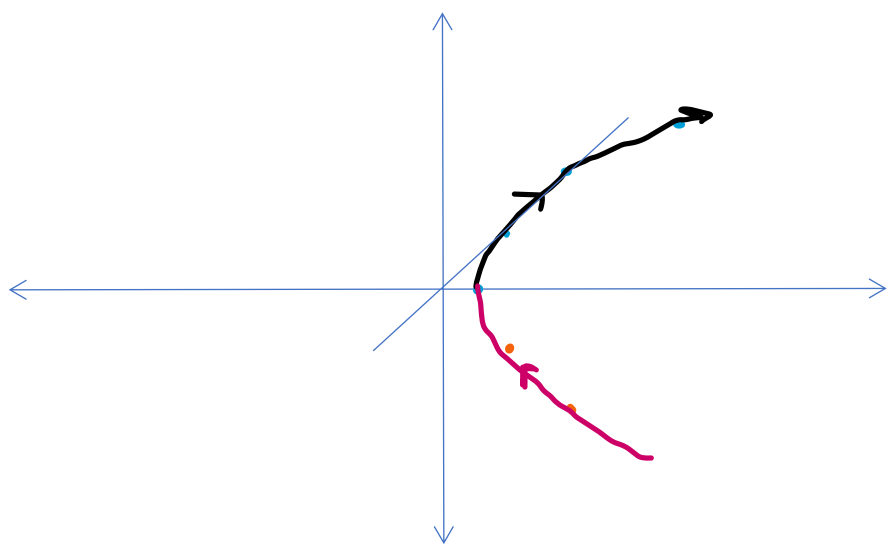
Set of parametric equations

$$x = x(t)$$
$$y = y(t)$$

The normal way we would plot these parametric equations is by selecting values of t (typically starting at t=0), and then plugging into the set of equation to obtain coordinate pairs. Then connect the dots.

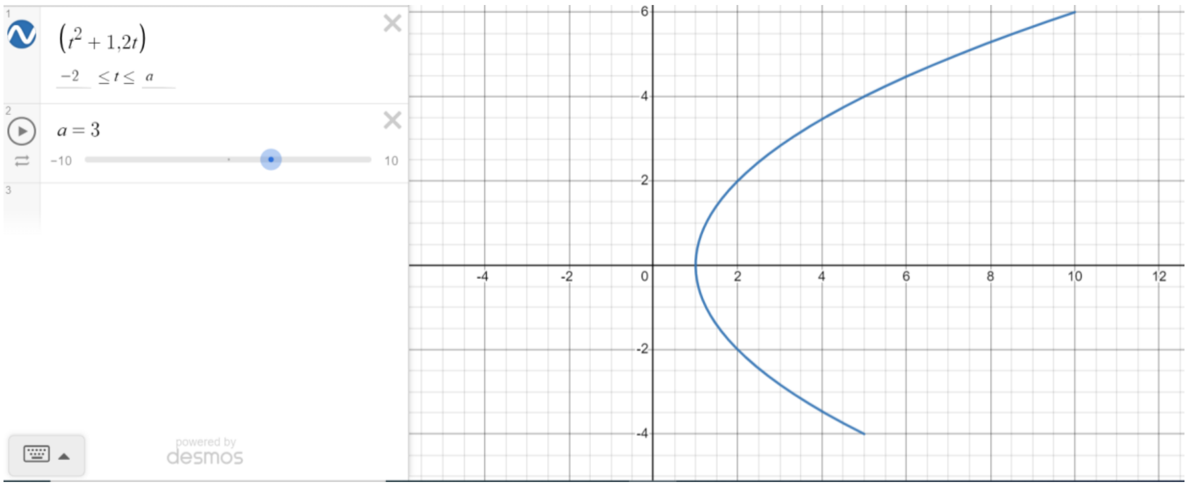
Example.

$$x(t) = t^2 + 1$$
$$y = 2t$$



$$t = 0 \to (1,0)$$
$$t = 1 \to (2,2)$$
$$t = 2 \to (5,4)$$
$$t = 3 \to (10,6)$$
$$t = -1 \to (2,-2)$$
$$t = -2 \to (5,-4)$$

Use an arrow to indicate the orientation of a curve: the direction of motion in which t is increasing.
We can use technology to do this.



To switch parametric equations back to rectangular coordinates (in terms of x and y only), solve for one of the t's (one of the parameters), and then substitute back in. (Caveat will come later.)

$$y = 2t$$
$$\frac{y}{2} = t$$

Substitute that back into the x equation.

$$x = \left(\frac{y}{2}\right)^2 + 1 = \frac{y^2}{4} + 1$$
$$x = \frac{y^2}{4} + 1$$

Aside about converting to rectangular.

$$x = \sin^2 t + 1$$
$$y = 2\sin t$$

If I do solve all the way for t, then I get $t = \arcsin\frac{y}{2}$. Then I would have to simplify. That's too much work. In a case like this, stop when you get to $\frac{y}{2} = \sin t$, because $\sin t$ is already in the other expression.

Don't leave me with $\sin(\arcsin anything)$ or $e^{\ln(anything)}$. Reduce all the way.
Solve for what the equations have in common.

What if we have a curve in x and y and we want to parametrize it.
If you can solve the equation for one variable (and not lose any information): either y is a function of x or x is a function of y.
Replace independent variable in the equation with your parameter (t), and then let the independent variable be the parameter.

$$y = e^x + x^2$$

Let $x = t$

$$y = e^t + t^2$$

You can pick any function of t to be the "parameter"

You could say $x = t^3$

$$y = e^{t^3} + t^6$$

If you don't have a particular reason to select a function of t as your parameter, then just stick with t.

Sometimes parametric sets of equations are represented as a list of separate equations (as we've been doing so far), or they can be listed as a vector.

$$r(t) = \langle x(t), y(t) \rangle = \langle t^2 + 1, 2t \rangle$$

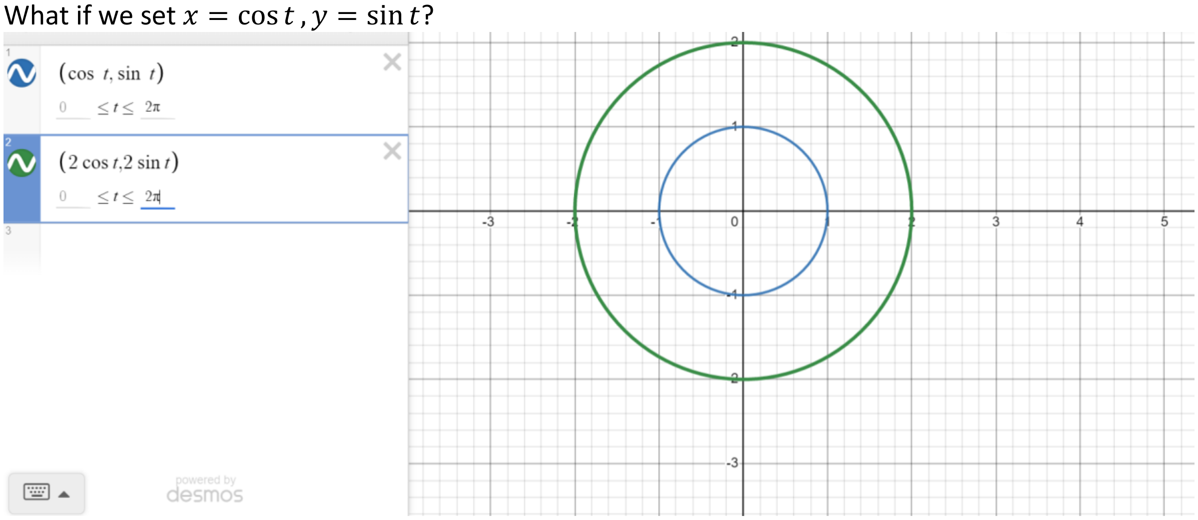
Here, we can just think of this as a notation for thinking of the set of parametric equations as a single object.

Special cases (not functions)

Circle. It is not a function. $x^2 + y^2 = 1$

$$\cos^2 t + \sin^2 t = 1$$

What if we set $x = \cos t, y = \sin t$?



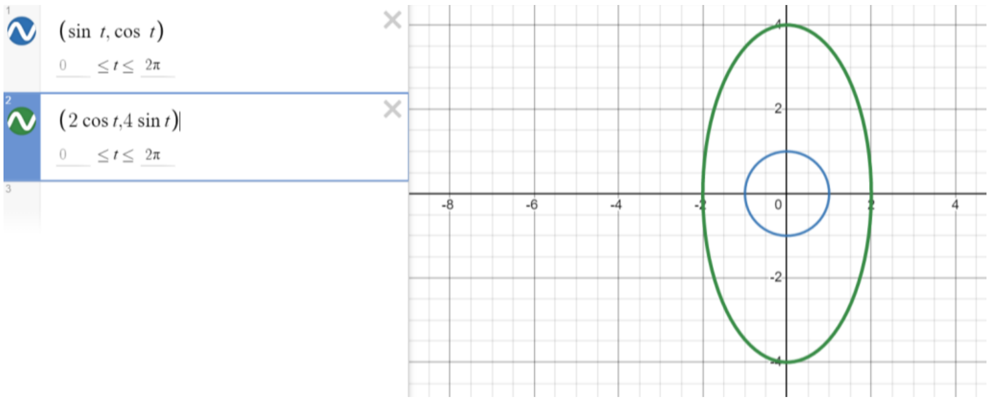
In general, to parameterize a circle, use

$$x(t) = r\cos t$$
$$y(t) = r\sin t$$

The orientation of the circle is counterclockwise. But if you switch sine and cosine, you can still get the whole circle, but you will start on the positive y-axis instead of the positive x-axis, and the orientation will be clockwise instead of counterclockwise.

We can extend this to get an ellipse. In an ellipse, the "radii" in different directions are different sizes.

$$x(t) = a\cos t$$
$$y(t) = b\sin t$$



Hyperbola

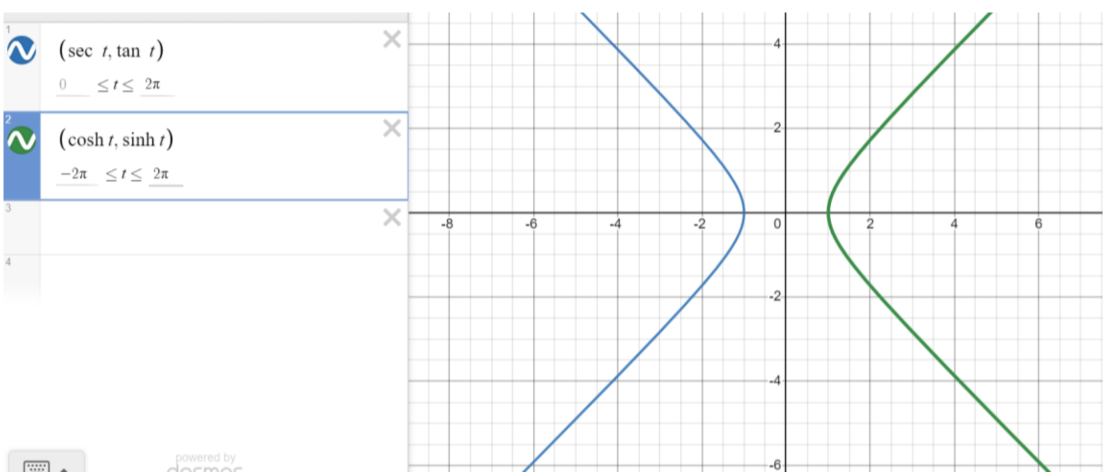
The general form of a hyperbola is $x^2 - y^2 = 1$

Alternative 1: start with $\tan^2 t + 1 = \sec^2 t \to \sec^2 t - \tan^2 t = 1$

$$x(t) = a\sec t$$
$$y(t) = b\tan t$$

Alternative 2: start with $\cosh^2 t - \sinh^2 t = 1$

$$x(t) = a\cosh t$$
$$y(t) = b\sinh t$$



Calculus of Parametric Equations.

We can take the derivative of parametric equations basically the same way we did of regular functions, but with respect to the parameter variable t.

$$r(t) = \langle x(t), y(t)\rangle$$
$$r'(t) = \langle x'(t), y'(t)\rangle$$

Earlier example

$$r(t) = \langle t^2 + 1{,}2t \rangle$$
$$r'(t) = \langle 2t, 2 \rangle$$

$$x'(t) = 2t$$
$$y'(t) = 2$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2}{2t} = \frac{1}{t}$$

Say I'm at the point where t=1, (2,2), then I know the slope of the tangent line is $\frac{dy}{dx} = \frac{1}{1} = 1$
The trick is that the problem may give you the coordinate in (x,y) and your derivative formula may be in terms of t. So you'll have to find the value of t that makes the point in the original set of equations.

Given the parametric equations $x = t^2 + 1, y = 2t$, find the slope of the tangent line at (2,2).

Set each coordinate equal to the appropriate equation to find t. $2 = 2t \to t = 1$, then check in the x-equation to make sure it works.

You could $2 = t^2 + 1 \to t^2 = 1$ which is either +1 or -1. You'll need to check the second equation to determine which one is the right one.

T cannot be different values in different coordinates of the same point. The coordinate pair must always use the same value of t.

The second derivative:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

Example. $\frac{dy}{dx} = \frac{1}{t}, \frac{dx}{dt} = 2t$

$$y'' = \frac{d^2y}{dx^2} = \frac{\left(-\frac{1}{t^2}\right)}{2t} = -\frac{1}{2t^3}$$

Area under a parametric curve

$$\int_a^b y(x)dx = \int_c^d y(t)x'(t)dt$$

Arc length of a parametric curve

$$s = \int_a^b \sqrt{1 + [f'(x)]^2}dx$$

For a standard rectangular curve.

$$s = \int_c^d \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Example.
Find length of arc for $r(t) = \langle 3\cos t\,, 3\sin t\rangle$ (a circle of radius 3), on the interval $[0,2\pi]$

$$s = \int_0^{2\pi} \sqrt{(-3\sin t)^2 + (3\cos t)^2} dt = \int_0^{2\pi} \sqrt{9\sin^2 t + 9\cos^2 t}\, dt = \int_0^{2\pi} \sqrt{9(\sin^2 t + \cos^2 t)} dt =$$

$$\int_0^{2\pi} 3dt = 6\pi$$

$$C = 2\pi r = 2\pi(3) = 6\pi$$

If you are doing arclength and you get a zero… check for needing absolute values. You may need to use symmetry.

Surface area for a solid of revolution.

$$S = \int_a^b r(x)\sqrt{1 + [f'(x)]^2} dx$$

In parametric form, we swap out the arc length as we did above, and the radius of rotation is either x(t) or y(t)

$$S_{around_x} = \int_c^d y(t)\sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dy}\right)^2}\, dt$$

$$S_{around_y} = \int_c^d x(t)\sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dy}\right)^2}\, dt$$

Horizontal and Vertical tangents:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

When is a tangent line horizontal? When the $\frac{dy}{dx} = 0$… so this will be zero when $\frac{dy}{dt} = 0$
When is the tangent line vertical? When the $\frac{dy}{dx}$ is undefined… that means that $\frac{dx}{dt} = 0$ (when $\frac{dy}{dt}$ is undefined).

Next time, we will cover polar coordinates, and calculus with polar equations and conic sections.