7/25/2022

Brief Introduction to Differential Equations (Chapter 4)

This material is not in MyOpenMath, so everyone needs to complete the Written Homework #9 in Canvas.

What is differential equations?
A differential equation is an equation that contain both a function and one or more of its derivatives.

$$y' = ky$$

The solution to this equation is a function which satisfies the relationship.

$$y = e^{kx}$$
$$y' = ke^{kx} = k(y)$$

Terminology
First order differential equation contains only the function and the first derivative.
Second order equation contains up the second derivative.
And so on.

All the examples are ordinary differential equations (they involve one independent variable). A partial differential equation contains partial derivatives (which you'll learn about in Calc III).

Differential equations arise a lot in physics. Say that you are balancing forces. You have gravity balanced with air resistance. Or in a spring, you have forces that depend on position, on damping (which depends on velocity), and on gravity (depends on acceleration).

Checking a solution.

$$y' - 3y = 6x + 4$$
$$y = 2e^{3x} - 2x - 2$$

$$y' = 6e^{3x} - 2$$

$$6e^{3x} - 2 - 3(2e^{3x} - 2x - 2) = 6e^{3x} - 2 - 6e^{3x} + 6x + 6 = 6x + 4$$

Checks out.

Example of solving a simple differential equation.

$$y' = 4x + 3$$

Integrate to find the solution when there is no other y in the problem.

$$\int y' = y = \int 4x + 3\, dx = 2x^2 + 3x + C$$

There is a family of solutions to this differential equation. To find the value of the constant we need to supply information about one point on the curve. If this point is at x=0, it's called an initial condition (boundary condition).

$$y(0) = 3$$

$$2(0)^2 + 3(0) + C = 3$$
$$C = 3$$

$$y = 2x^2 + 3x + 3$$

This function satisfies both the initial condition and the differential equation.

Initial value problems (they come with initial values).
Sometimes they give you the point as pair of coordinates (0,3).

You may need integration techniques for some of these kind of problems. Depending on how complex the function is.

Visualizing differential equations: Slope fields, direction fields.

The kind of differential equation we want to first think about visualizing are called autonomous equations.

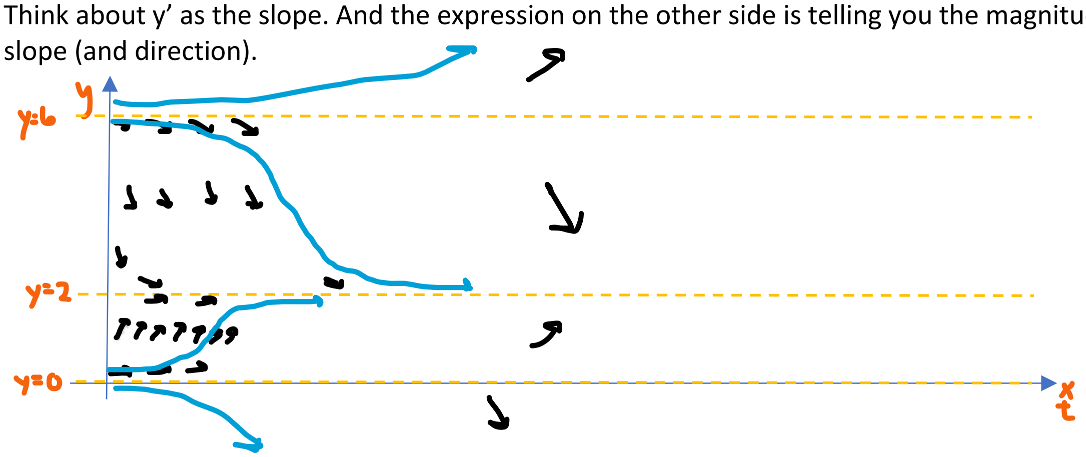
Basically an autonomous includes the derivative and its function but no terms that contain the independent variable.

$$y' = ky$$
$$y' = ky(y-2)(y-6)$$

These are examples of autonomous equations.

Typically, when we plot direction fields, we need to worry about what both x and y are doing. But for an autonomous equation, the behavior doesn't depend on x (or t) at all.

Think about y' as the slope. And the expression on the other side is telling you the magnitude of the slope (and direction).

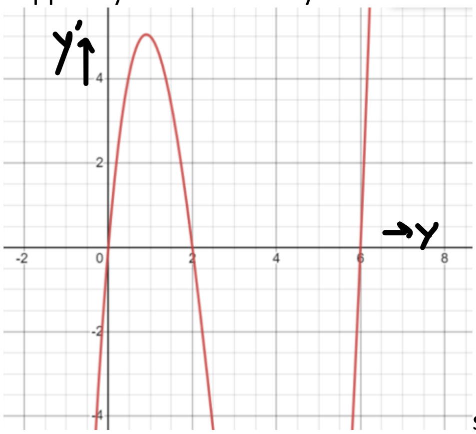


A trick for plotting these autonomous equations is to find out where the slope is zero. These are called the equilibria (-ium is singular).

$$y' = y(y-2)(y-6)$$
$$0 = y(y-2)(y-6)$$

Equilibria are going to be at y=0, y=2, y=6.

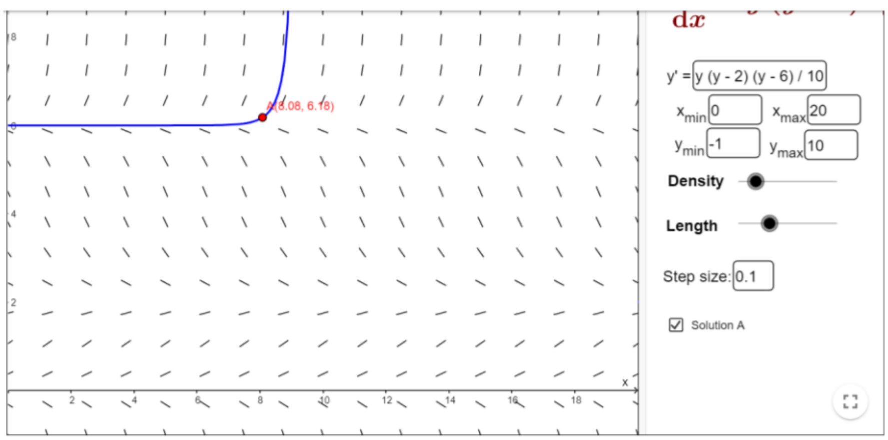
Suppose you start out at y=1. What is slope looking like? $y' = 1(-1)(-5) = 5$



sometimes this graph is called a phase plane.

Y=3, $y' = 3(3-2)(3-6) = 3(1)(-3) = -9$

https://www.geogebra.org/m/Pd4Hn4BR



Terminology:

In the population context, carrying capacity. A carrying capacity is a stable equilibrium. It attracts solutions. If you are above or below it, you tend to go toward that stable value.

Unstable equilibrium is a threshold. Below it you collapse to extinction. Above it you grow.

You can also have what are semi-stable equilibria. You are attracted on one side and repelled on the other.

The S-shaped curves in our slope field are called logistic curves.

In section 4.4, the problems there tend to deal with solutions from these kinds of functions, but they give you the solution functions. To solve them by hand, we would need partial fractions (to derive the solutions), so we won't actually do that. For use, be able to use the solution equation provided, and think about the curves holistically from the slope field.

If we don't have an autonomous equation, then the slope depends on both variables. Plotting them with technology is fine.

Euler's Method.
Euler's method is a numerical approach to solving differential equations, to approximate their solutions, rooted in the idea of linear approximations.

We start at some point. We estimate the slope (the general direction of change from that point). We move in that direction in a linear fashion. And then re-estimate and continue.

Initial starting point. y_0 is the initial starting point in y. x_0 is the initial starting point in x. the first slope is $y'(x_0, y_0) = m_1$. Then we estimate the new positions for $x_1 = x_0 + h$ (h is the step size). And $y_1 = m(h) + y_0$.
Then recalculate the slope, etc. to find the next point.

$$m_n = y'(x_{n-1}, y_{n-1})$$

$$x_{n+1} = x_n + h$$
$$y_{n+1} = m(h) + y_n$$

Suppose we want to estimate the value of $y(1)$, after starting at $(0,1)$ for the equation $y' = 3x + y$. Use step size h=0.1.
Sometimes they will tell you the number of steps (n=10 for instance). $h = \frac{b-a}{n}$, where a is the starting value for x, and b is the stopping value for x. $\frac{1-0}{10} = 0.1$.

$$m_1 = 3(0) + 1 = 1$$

$$x_1 = 0 + 0.1 = 0.1$$
$$y_1 = (1)(0.1) + 1 = 1.1$$

Step 2.

$$m_2 = 3(0.1) + 1.1 = 1.4$$

$$x_2 = 0.1 + 0.1 = 0.2$$
$$y_2 = 1.4(0.1) + 1.1 = 1.24$$

Continue.

One particular solution method is called separation of variables.

$$y' = 3e^x(y^2 + 1)$$

This works when it is possible to algebraically put all the x-variable terms on one side of the equation and all the y-variable terms on the other side of the equation (generally by multiplying or dividing).

$$\frac{dy}{dx} = 3e^x(y^2 + 1)$$

Divide by $y^2 + 1$ and multiply by dx.

$$\frac{dy}{y^2 + 1} = 3e^x dx$$

Integrate each side for the corresponding variables.

$$\int \frac{1}{y^2 + 1} dy = \int 3e^x dx$$

$$\arctan y = 3e^x + C$$

$$y = \tan(3e^x + C)$$

If we had initial conditions, we could solve for the constant.
Generally speaking, for these separation of variable problems tend be ugly and implicit. It is okay to leave as implicit functions. Do not feel obligated to solve for y at all.