7/18/2022

Constructing and using power series
Taylor Series

Note on the second exam… there is a change to the calendar.
We will meet for lecture on Wednesday to finish the Chapter 7 material, if necessary, and do review (which we won't have time for today). I will open the exam when the session finishes. Then students will have until Friday to complete the test.

Schedule resumes as normal on Monday.

Constructing Power Series
Start with the formula for the sum of a geometric series

$$\sum_{n=0}^{\infty} a(r^n) = \frac{a}{1-r}$$

For $|r| < 1$

$$\sum_{n=0}^{\infty} a(x^n) = \frac{a}{1-x}$$

For example.

$$f(x) = \frac{3x^2}{1-2x}$$

What is r? What is a?

$$r = 2x, a = 3x^2$$

$$\frac{3x^2}{1-2x} = \sum_{n=0}^{\infty} 3x^2(2x)^n = \sum_{n=0}^{\infty} 3x^2(2^n x^n) = \sum_{n=0}^{\infty} 3(2^n)x^{n+2}$$

We would have to test for convergence (like we did last time) to see where the function is defined.

The constant in the denominator must be 1 (we may have to rescale if the constant is not 1). Also the formula has a minus sign in it, so if the denominator has a + sign, then we have to rewrite so that we have a minus sign.

Example.

$$f(x) = \frac{x}{7+x^2} = \frac{x}{7-(-x^2)} = \frac{x\left(\frac{1}{7}\right)}{\frac{1}{7}(7+x^2)} = \frac{\frac{x}{7}}{1+\frac{x^2}{7}} = \frac{\frac{x}{7}}{1-\left(-\frac{x^2}{7}\right)}$$

We want to write this as a power series.

$$\sum_{n=0}^{\infty} a(r^n) = \frac{a}{1-r}$$

$$r = \left(-\frac{x^2}{7}\right), a = \frac{x}{7}$$

$$\frac{x}{7+x^2} = \sum_{n=0}^{\infty} \left(\frac{x}{7}\right)\left(-\frac{x^2}{7}\right)^n = \sum_{n=0}^{\infty} \frac{x}{7}(-1)^n(x^{2n})\left(\frac{1}{7}\right)^n = \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{7}\right)^{n+1} x^{2n+1}$$

But now what if we have a function like $f(x) = \frac{x}{(1+4x)^2}$?

$$\sum_{n=0}^{\infty} a(x^n) = \frac{a}{1-x}$$

The solution here (if the denominator is raised to a power) is to take derivatives of our original formula.

$$f(x) = \frac{a}{1-x} = a(1-x)^{-1} = \sum_{n=0}^{\infty} a(x^n)$$

$$f'(x) = (-1)a(1-x)^{-2}(-1) = a(1-x)^{-2} = \frac{a}{(1-x)^2} = \sum_{n=1}^{\infty} an(x^{n-1}) = \sum_{k=0}^{\infty} a(k+1)(x^k)$$

$$f''(x) = -2a(1-x)^{-3}(-1) = \frac{2a}{(1-x)^3} = \sum_{n=2}^{\infty} an(n-1)(x^{n-2}) = \sum_{k=0}^{\infty} a(k+2)(k+1)(x^k)$$

$$f'''(x) = \frac{6a}{(1-x)^4} = \sum_{n=3}^{\infty} an(n-1)(n-2)(x^{n-3}) = \sum_{k=0}^{\infty} a(k+3)(k+2)(k+1)(x^k)$$

Now, the issue that we usually start power series with n=0, not n=3.

Re-indexing. Replace n-1=k implies n=k+1

$$\sum_{n=1}^{\infty} an(x^{n-1}) = \sum_{k+1=1}^{\infty} a(k+1)(x^{k+1-1}) = \sum_{k=0}^{\infty} a(k+1)(x^k)$$

Replace n-2 = k or n=k+2

$$\sum_{n=2}^{\infty} an(n-1)(x^{n-2}) = \sum_{k+2=2}^{\infty} a(k+2)(k+2-1)(x^{k+2-2}) = \sum_{k=0}^{\infty} a(k+2)(k+1)(x^k)$$

Replace n-3=k or n=k+3

$$\sum_{n=3}^{\infty} an(n-1)(n-2)(x^{n-3}) = \sum_{k+3=3}^{\infty} a(k+3)(k+3-1)(k+3-2)\left(x^{k+3-3}\right)$$

$$= \sum_{k=0}^{\infty} a(k+3)(k+2)(k+1)\left(x^{k}\right)$$

For whatever power you have over the entire denominator, as long as it's an integer, you can generate a power series to suit it by taking derivatives of the based geometric series formula.

Example: we have a function like $f(x) = \frac{x}{(1+4x)^3}$. We want to make a power series of it.

The formulas we need:

$$\frac{2a}{(1-r)^3} = \sum_{k=0}^{\infty} a(k+2)(k+1)\left(r^{k}\right)$$

What are a and r?

$$r = -4x$$
$$2a = x \to a = \frac{x}{2}$$

$$f(x) = \frac{x}{(1+4x)^3} = \sum_{k=0}^{\infty} \frac{x}{2}(k+2)(k+1)(-4x)^k = \sum_{k=0}^{\infty} (-1)^k\left(4^k\right)\frac{1}{2}(k+2)(k+1)x^{k+1}$$

$$\frac{4^k}{2} = \frac{2^{2k}}{2} = 2^{2k-1}$$

$$\sum_{k=0}^{\infty} (-1)^k 2^{2k-1}(k+2)(k+1)x^{k+1}$$

Suppose I want to differentiate $f(x) = \frac{x}{(1+4x)^3}$?
I would need a quotient with an embedded chain rule.

Or what if I wanted to integrate it?
I would need to partial fractions with three terms. And then u-sub to integrate the results.

But in the form of a power series, differentiating is just a power rule. And integrating is also just a power rule.

$$f(x) = \frac{x}{(1+4x)^3} = \sum_{k=0}^{\infty} (-1)^k 2^{2k-1}(k+2)(k+1)x^{k+1}$$

$$f'(x) = \sum_{k=0}^{\infty} (-1)^k 2^{2k-1}(k+2)(k+1)(k+1)x^{k}$$

$$\int f(x)dx = \int \frac{x}{(1+4x)^3} dx = \int \sum_{k=0}^{\infty} (-1)^k 2^{2k-1}(k+2)(k+1)x^{k+1}\, dx$$

$$= \sum_{k=0}^{\infty} \int (-1)^k 2^{2k-1}(k+2)(k+1)x^{k+1} dx = \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k-1}(k+2)(k+1)x^{k+2}}{k+2} + C$$

$$= \sum_{k=0}^{\infty} (-1)^k 2^{2k-1}(k+1)x^{k+2} + C$$

So, if we have $f(x) = \frac{3x^2}{2x-1} = -\frac{3x^2}{-1(2x-1)} = -\frac{3x^2}{1-2x}$

Two special cases of functions that are not themselves rational, but which have rational derivatives that we can use this method for generating power series for them.

The two function are $f(x) = \ln(x), g(x) = \arctan(x)$

$$f'(x) = \frac{1}{x}, g'(x) = \frac{1}{1+x^2}$$

The general procedure we use to generate a power series is:

1) Take the derivative to obtain a ration function
2) Make a power series of the rational expression
3) Integrate the power series to obtain the power series for the original function.

Let's find a power series for the ln(x) function.

1) $f(x) = \ln(x), f'(x) = \frac{1}{x}$

   Rewrite to shift the center.

$$f'(x) = \frac{1}{x-1+1} = \frac{1}{1+(x-1)} = \frac{1}{1-(1-x)}$$

$$a = 1, r = (1-x) = (-1)(x-1)$$

2) Make the power series.

$$\sum_{n=0}^{\infty} a(r^n) = \frac{a}{1-r}$$

$$\frac{1}{x} = \frac{1}{1-(1-x)} = \sum_{n=0}^{\infty} 1\left((-1)(x-1)\right)^n = \sum_{n=0}^{\infty}(-1)^n(x-1)^n$$

3) Do the antiderivative to get back to the original function.

$$\ln(x) = \sum_{n=0}^{\infty} \frac{(-1)^n(x-1)^{n+1}}{n+1} + C$$

When x=1, what is ln(1)? The constant is 0 because the center of the function at x=1, is also 0.

$$\ln(x) = \sum_{n=0}^{\infty} \frac{(-1)^n(x-1)^{n+1}}{n+1}$$

Taylor Series.
A method for generating power series for functions that cannot be expressed as rational functions (or its derivatives). This includes things like trig functions, and exponential functions, etc. Can also include fractional exponents of x.
When the Taylor series is centered at 0, this is often referred to as a Maclaurin series. But they both follow the same derivations other than the center value.

The Taylor uses derivatives of the original function to approximate it. The first term is the value of the original function at the center of our expansion. $f(c)$

The next term in the Taylor series is based on the linear approximation to the curve at that point (based on the first derivative). $f'(c)(x-c)$

We continue in this process taking derivatives, approximate the value and centered at the point x=c.
$\frac{f''(c)(x-c)^2}{2}$

Continue in this vein…

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)(x-c)^n}{n!} = \frac{f(c)(x-c)^0}{0!} + \frac{f'(c)(x-c)^1}{1!} + \frac{f''(c)(x-c)^2}{2!} + \frac{f'''(c)(x-c)^3}{3!} + \cdots$$

Suppose I want to generate a Taylor series for $f(x) = e^x$
Use all n=6, c=0, then generate a general formula for the Taylor series.

n	$n!$	$f^{(n)}(x)$	$f^{(n)}(c)$	$(x-c)^n$	$\frac{f^{(n)}(c)(x-c)^n}{n!}$
0	1	e^x	1	1	1

1	1	e^x	1	x	x
2	2	e^x	1	x^2	$\frac{x^2}{2}$
3	6	e^x	1	x^3	$\frac{x^3}{6}$
4	24	e^x	1	x^4	$\frac{x^4}{24}$
5	120	e^x	1	x^5	$\frac{x^5}{120}$
6	720	e^x	1	x^6	$\frac{x^6}{720}$

Taylor polynomial is an approximation of the Taylor series, $P_6(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720}$.

The general formula for the Taylor (Maclaurin) series for e^x centered at x=0, is $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

What if I wanted to create a power series for e^{x^2}? Rather than generating the formula from scratch, I can just replace x with x^2 in my e^x series.

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$$

Now, if I want to integrate this function, I can. I can't integrate the original version, but I can integrate the power series.

Error term.

The error for a Taylor series is based on the next term in the series.

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1} = \frac{\max|f^{n+1}(x)|\,(x-c)^{n+1}}{(n+1)!}$$

The max value or the expression $f^{(n+1)}(z)$ is interpreted as the largest magnitude value of this derivative on some interval of interest (say for an integral) or for some range of values like ± 1 from the center.

Suppose I want to generate a Taylor series for $f(x) = e^x$
Use all n=4, c=0, then generate a general formula for the Taylor series. Then calculate the error at x=1/2.

For the most part, we proceed as we did before.

n	$n!$	$f^{(n)}(x)$	$f^{(n)}(c)$	$(x-c)^n$	$\frac{f^{(n)}(c)(x-c)^n}{n!}$
0	1	e^x	1	1	1
1	1	e^x	1	x	x

2	2	e^x	1	x^2	$\frac{x^2}{2}$
3	6	e^x	1	x^3	$\frac{x^3}{6}$
4	24	e^x	1	x^4	$\frac{x^4}{24}$
5	120	e^x	1	x^5	$\frac{x^5}{120}$

Taylor polynomial is an approximation of the Taylor series, $P_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$.

What about the error. The $f^{(5)}(x) = e^x$. Approximating the error at x=1/2.

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1} = \frac{\max|f^{n+1}(x)|\,(x-c)^{n+1}}{(n+1)!}$$

I want to think about the interval (since we are centered at x=0) $\left(-\frac{1}{2}, \frac{1}{2}\right)$. Take the distance from the center as your radius. Where is e^x the largest (magnitude) on this interval? It's gonna be at x=1/2.

$$R_n(x) = \frac{e^{\frac{1}{2}}\left(\frac{1}{2} - 0\right)^5}{5!} \approx \frac{e^{\frac{1}{2}}\left(\frac{1}{2}\right)^5}{120} \approx 4.29 \times 10^{-4} \approx 0.000429 \ldots$$

What can we do with Taylor series besides differentiating and integrating?

If we want to shift the center of the function, you may need to recalculate the coefficients.

Suppose you replace a function centered at 0 using x with $x - 3$, then the best approximation is still going to be at x=0.

Suppose we want to generate the power series for $f(x) = \sin(\sqrt{x})$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\sin(\sqrt{x}) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^{\frac{1}{2}})^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+\frac{1}{2}}}{(2n+1)!}$$

Suppose I want to generate a power series for $f(x) = x^3 e^{x^2}$

$$f(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}(x^3) = \sum_{n=0}^{\infty} \frac{x^{2n+3}}{n!}$$

Trickier, what if I wanted to generate a power series for $f(x) = e^x \sin(x)$?

$$e^x \sin(x) = \left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)\left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}\right) =$$

$$\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots\right)\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right)$$

Can't just multiply the general term by each other. I have to FOIL. Figure out how many powers you want, and then stop FOILing when your terms exceed that power.

Suppose I want to stop at x^3

$$\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots\right)\left(x - \frac{x^3}{6} + \cdots\right)$$

$x - \frac{x^3}{6} + x^2 + \frac{x^3}{2}$... terms up to x^3, and so just simplify and proceed with the problem.

We can still talk about division.
Applications (in say differential equations).

Review for the exam on Wednesday.

$$\sum_{n=1}^{\infty} \frac{n!}{n^n} x^n$$

$$\lim_{n\to\infty} \frac{(n+1)!\, x^{n+1}}{(n+1)^{n+1}} \times \frac{n^n}{n!\, x^n} = \lim_{n\to\infty} \frac{(n+1) n!\, x^n x}{(n+1)^n (n+1)} \times \frac{n^n}{n!\, x^n} = \lim_{n\to\infty} \frac{n^n x}{(n+1)^n}$$

$$= x \lim_{n\to\infty} \frac{n^n}{(n+1)^n} = \frac{x}{e}$$

$$\lim_{n\to\infty} \frac{(n+1)^n}{n^n} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$$