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Ratio & Root Tests (5.6)
Intro to Power Series

Ratio Test
Suppose you have an infinite series $\sum a_n$ and you want to determine if it converges. If the $|a_n| \geq |a_{n+1}|$, then if the $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$ then the series converges. If $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$ the series diverges. If the $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$ the test is inconclusive. (you need to try another test.)

Both of these tests are something of an extension of the geometric series test.

$$\sum \left(\tfrac{1}{3}\right)^n \text{ vs. } \sum n \left(\tfrac{1}{3}\right)^n$$

Root Test
Suppose you have an infinite series $\sum a_n$ and you want to determine if it converges. If the $|a_n| \geq |a_{n+1}|$, then if $\lim\limits_{n\to\infty} \sqrt[n]{a_n} < 1$, then the series converges. If $\lim\limits_{n\to\infty} \sqrt[n]{a_n} > 1$ then the series diverges. And if the $\lim\limits_{n\to\infty} \sqrt[n]{a_n} = 1$, then the test is inconclusive (you need to try another test).

$$\lim_{n\to\infty} \sqrt[n]{n} = 1$$

Factorials grow faster than any exponential.

Examples.

$$\sum n \left(\frac{1}{3}\right)^n$$

(we can assume the series starts at either 0 or 1)

We want the absolutely value of the sequence of terms to decrease, and that is okay as long as it decreasing everywhere after a finite number of terms. You can test this with the derivative or by graphing.

$$\lim_{n\to\infty} \frac{(n+1)\left(\frac{1}{3}\right)^{n+1}}{n\left(\frac{1}{3}\right)^n} = \left(\lim_{n\to\infty} \frac{n+1}{n}\right)\left(\lim_{n\to\infty} \frac{\left(\frac{1}{3}\right)^{n+1}}{\left(\frac{1}{3}\right)^n}\right) = \left(\lim_{n\to\infty} 1 + \frac{1}{n}\right)\left(\lim_{n\to\infty} \frac{\left(\frac{1}{3}\right)^n \left(\frac{1}{3}\right)^1}{\left(\frac{1}{3}\right)^n}\right) = (1)\left(\frac{1}{3}\right) = \frac{1}{3}$$

This ratio is less than 1, and so the series converges.

We can do this same problem by the root test.

$$\lim_{n\to\infty} \sqrt[n]{n\left(\frac{1}{3}\right)^n} = \left(\lim_{n\to\infty} \sqrt[n]{n}\right)\left(\lim_{n\to\infty} \sqrt[n]{\left(\frac{1}{3}\right)^n}\right) = (1)\left(\frac{1}{3}\right) = \frac{1}{3}$$

This limit is less than 1, and so the series converges.

Use the root test if you have terms in both the numerator and denominator that have powers of n. Particularly if you have n^n.

Use the ratio test especially if you have any factorials in the expression. (Factorial and the nth root do not “play” well together.)

Example.

$$\sum_{n=0}^{\infty} \frac{ne^{2n}}{n!}$$

Use the ratio test (with the factorial) to test to see if the series converges.

$$\lim_{n\to\infty} \frac{(n+1)e^{2(n+1)}}{(n+1)!} \times \frac{n!}{ne^{2n}} = \lim_{n\to\infty} \frac{(n+1)e^{2n}e^2}{(n+1)n!} \times \frac{n!}{ne^{2n}} = \lim_{n\to\infty} \frac{e^2}{n} = 0$$

This limit is less than 1, and so the series converges.

We want to consolidate our knowledge of series tests. Which tests to use when.

Divergence Test (Nth term test)
Alternating Series Test (errors)
Integral Test (errors)
Geometric Series Test (exact sums)
Telescoping Series Test (exact sums)
P-series Test
Direct Comparison Test
Limit Comparison Test
Ratio Test
Root Test

Best overall strategy is to start with the simplest test for the problem you have.
See Series Test handout.

Power Series
These are infinite series that contain a variable (x) raised to a power (thus the name), and what we want to start considering is if we have a power series, does it converge and if not everywhere, where does it converge (interval, but in some limited cases it might be a point).

Generally, when we test for convergence, we pretend that x is a fixed value. And we perform regular series tests to determine convergence. And then, if x survives into our solution (limit contains x), then we will solve for x from there.

We want to look at an example.

$$f(x)=\sum_{n=1}^{\infty}\frac{nx^n}{2^{n+1}}$$

Does the series converge, and if not everywhere, where?

Perform a ratio test on the series, because this will give us a condition on convergence.

$$\lim_{n\to\infty}\left|\frac{(n+1)x^{n+1}}{2^{n+2}}\times\frac{2^{n+1}}{nx^n}\right|=\lim_{n\to\infty}\left|\frac{(n+1)x^n x}{2^{n+1}(2)}\times\frac{2^{n+1}}{nx^n}\right|=\lim_{n\to\infty}\left|\frac{(n+1)x}{2}\times\frac{1}{n}\right|$$
$$=\left(\lim_{n\to\infty}\left|\frac{n+1}{n}\right|\right)\left(\lim_{n\to\infty}\left|\frac{x}{2}\right|\right)=\left|\frac{x}{2}\right|$$

Condition on convergence is limit has to be less than 1.

$$\left|\frac{x}{2}\right|<1$$

$$-1<\frac{x}{2}<1$$

$$-2<x<2$$

At least on the interval $(-2,2)$ will converge. But when x=-2, or x=2, the ratio test is inconclusive. So I also need to test the endpoints to see if they converge.

Test the endpoints:

$$f(-2)=\sum_{n=1}^{\infty}\frac{n(-2)^n}{2^{n+1}}=\sum_{n=1}^{\infty}\frac{n(-1)^n2^n}{2^{n+1}}=\sum_{n=1}^{\infty}\frac{(-1)^n n}{2}$$

Alternating series test. Does this converge? The limit is not zero. Therefore, it diverges.

$$f(2)=\sum_{n=1}^{\infty}\frac{n2^n}{2^{n+1}}=\sum_{n=1}^{\infty}\frac{n}{2}$$

Use Divergence test (nth term test). The limit is not zero, and so it diverges.

So the interval of convergence is $(-2,2)$.

Sometimes you may be asked to determine the radius of convergence.
If the interval of convergence is (a,b), then the radius of convergence is $\frac{b-a}{2}$.

If I had $(x-4)^n$ in the problem, then the interval is not centered at zero. In this case, the endpoint of the interval is not the radius.

Example.

$$\sum_{n=1}^{\infty} \frac{(-1)^n n(x-4)^n}{n!}$$

$$\lim_{n\to\infty} \left| \frac{(n+1)(x-4)^{n+1}}{(n+1)!} \times \frac{n!}{n(x-4)^n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)(x-4)^n(x-4)}{(n+1)(n!)} \times \frac{n!}{n(x-4)^n} \right| =$$

$$\lim_{n\to\infty} \left| \frac{x-4}{n} \right| = 0$$

The interval of convergence is all real numbers $(-\infty, \infty)$. And the radius of convergence is also infinity.

Example.

$$\sum_{n=1}^{\infty} n!\,(x-10)^n$$

Ratio test

$$\lim_{n\to\infty} \frac{(n+1)!\,(x-10)^{n+1}}{n!\,(x-10)^n} = \lim_{n\to\infty} (n+1)(x-10) = \begin{cases} \infty, & when\ x \neq 10 \\ 0, & when\ x = 10 \end{cases}$$

The interval of convergence is {10}.
The radius of convergence is 0.

Convergence of a power series.
Generally, use the ratio test (typically—if there are no factorials, the root test also works) to determine a condition on convergence that depends on x. The absolute value of your expression should be less than 1.

If your limit is a finite number and x,

- Solve for the interval of convergence (break out your absolute value into a two-sided interval).
- Make your coefficient of x=1 (give you the radius of convergence)
- Solve for x alone, to get the interval of convergence (use the formula if needed to get the radius)
- Check the endpoints for convergence – neither may work, one may work or both may work. If one or more works this changes the interval from a round bracket to a square bracket.

If your limit is 0, then the interval of convergence is all real numbers, and the radius is infinity. There are no endpoints, so you don't have to check them.

If your limit is infinity, check to see if there is any (single) value where the value of terms in the sequence are always 0. Then your interval of convergence will be a single point. {}, and the radius of convergence is 0.

Next week,
We are going to look at Power Series.

We are going to create power series. Convert known functions into power series representations using the geometric sum formula.

$$\sum_{n=0}^{\infty} a(r^n) = \frac{a}{1-r}$$

Derivatives of this rule to extend our ability to make power series.
Use what we did earlier today to test for convergence, and where.