7/11/2022

Alternating Series Test
Integral Test
P-Series Test
Comparison Tests – Direct and Limit

Alternating Series Test vs. Divergence Test

Divergence Test: if the limit of sequence (that is summed to get a series) if it does not go to zero, then the series diverges. If the limit does go to zero, we can't say anything.

Alternating Series Test: Sequence a_n is positive, and series is $\sum(-1)^n a_n$.
Example. $\sum \frac{(-1)^n}{n} = \left(-1+\frac{1}{2}\right)+\left(-\frac{1}{3}+\frac{1}{4}\right)+\left(-\frac{1}{5}+\frac{1}{6}\right)-\cdots\left(-\frac{1}{2}\right)+\left(-\frac{1}{12}\right)+\left(-\frac{1}{30}\right)+.$
If the $\lim_{n\to\infty} a_n = 0$ then the series converges. If the limit doesn't go to zero, then the series diverges.

Alternating Series Test has an error estimate.
If you take the sum of n terms of the alternating series, then the error on the nth term is the n+1st term.

Using the alternating harmonic series $\sum \frac{(-1)^n}{n} = \left(-1+\frac{1}{2}\right)+\left(-\frac{1}{3}+\frac{1}{4}\right)+\left(-\frac{1}{5}+\frac{1}{6}\right)+\left(-\frac{1}{7}+\frac{1}{8}\right)+$ $\left(-\frac{1}{9}+\frac{1}{10}\right)\ldots = \left(-\frac{1}{2}\right)+\left(-\frac{1}{12}\right)+\left(-\frac{1}{30}\right)+\left(-\frac{1}{56}\right)+\left(-\frac{1}{90}\right) = -\frac{1627}{2520} \approx -0.6456349206\ldots \pm \frac{1}{11}$

The true value of the sum is between $(-0.6456349206-\frac{1}{11}, -0.6456349206+\frac{1}{11})$.

If you wanted an error on the sum to be less than $10^{-4} = 0.0001$.
Set the $n+1^{st}$ term to the error.

$$\frac{1}{n+1} = 10^{-4}$$

$$n+1 = 10^4$$

$n = 9999$ terms to guarantee that the sum was within 0.0001 of the true value.

N must be an integer, so if you get a decimal, you will need to round up.

Example.

$$\sum \frac{(-1)^n e^{-n}}{n}$$

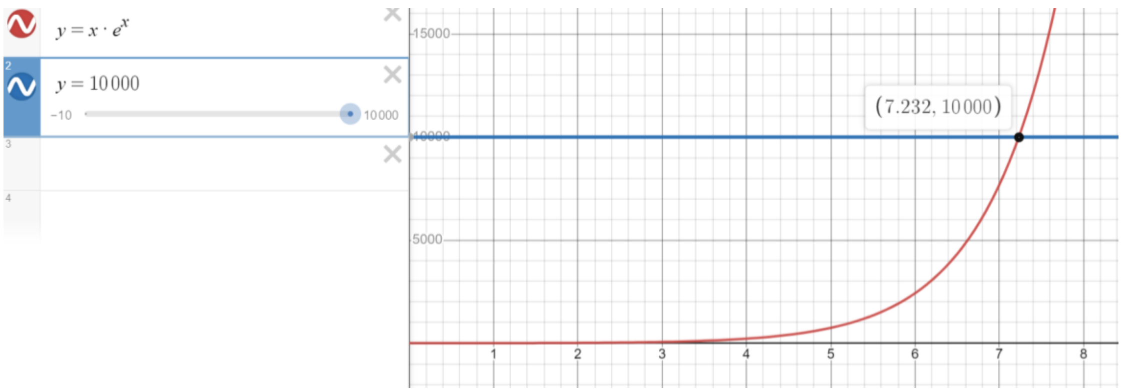
$$\lim_{n\to\infty} \frac{e^{-n}}{n} = \lim_{n\to\infty} \frac{1}{ne^n} = 0$$

How many terms do we need to get an error less than 10^{-4}?

$$\frac{1}{Ne^N} = 10^{-4}$$

$$10^4 = Ne^N$$

N=7.23… $n + 1 = 7.23…$,$n + 1 = 8$
Add up the first 7 terms.



Conditional Convergence
If an alternating series converges because it is alternating, but does not converge if it is not alternating (we take the absolute/ or all positive terms), then the series is said to converge conditionally. If the convergence does not depend on the fact that it is alternating, then the series converges absolutely.

Integral Test
Consider a series $\sum a_n = \sum f(n)$, if $\int_0^\infty f(x)dx$ converges, then the series converges. If the integral diverges, then the series diverges. Note that a_n is assumed to be greater than zero here.

Consider the alternative harmonic series.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

$(-1)^x$ is not a continuous function, so we integrate the $|a_n|$ in this case.

We are test the series to see if it converges absolutely. Put another way, to test to see if $\sum_{n=1}^{\infty} \frac{1}{n}$ converges or diverges.

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots = diverges$$

Use the integral test to show this diverges:

$$\int_1^\infty \frac{1}{x} dx = \ln(x)|_1^\infty = \lim_{b \to \infty} \ln(b) - \ln(1) = \infty$$

Think of the sums of the terms as rectangles with width 1, sum of the rectangles is the area (approximately). So if the integral is finite, then series is finite. If the integral is infinite, then the series diverges.

If the integral converges, the sum of the series is not necessarily the same as the value of the integral.

The alternating harmonic series is conditionally convergent. It converges because it is alternating. It fails to converge without alternating.

$$\sum (-1)^n n e^{-n}$$

Is it at least conditionally convergent?

$$\lim_{n\to\infty} ne^{-n} = \lim_{n\to\infty} \frac{n}{e^n} = \lim_{n\to\infty} \frac{1}{e^n} = 0$$

Is it going to converge absolutely?
Use integral test

$$\int_0^\infty xe^{-x}dx = -xe^{-x} + \int_0^\infty e^{-x}dx = [-xe^{-x} - e^{-x}]_0^\infty =$$

$$u = x, du = e^{-x}dx, du = dx, v = -e^{-x}$$

$$\lim_{b\to\infty}\left(-be^{-b} - e^{-b}\right) - (0-1) = 1$$

The integral converges, and therefore, so does the series. This series is absolutely convergent. Converges regardless of whether it is alternating or not.

If I know a series converges, and I want to know how many terms I will need to guarantee that the series has less than a given error size, I can use the integral test to find the number of terms needed.

I'm also going to use N=n+1

Suppose I add terms of a series $\sum_{k=1}^{n} a_k$ (add up the first k terms of the series). The rest of the series can be written as $\sum_{k=n+1}^{\infty} a_k$. And this infinite series is approximately $\int_N^\infty f(k)dk$, and this end of the series is an estimate of the error on the earlier part of the sum.

$$\sum_{k=1}^{\infty} a_k \approx \sum_{k=1}^{n} a_k + \int_N^\infty f(k)dk$$

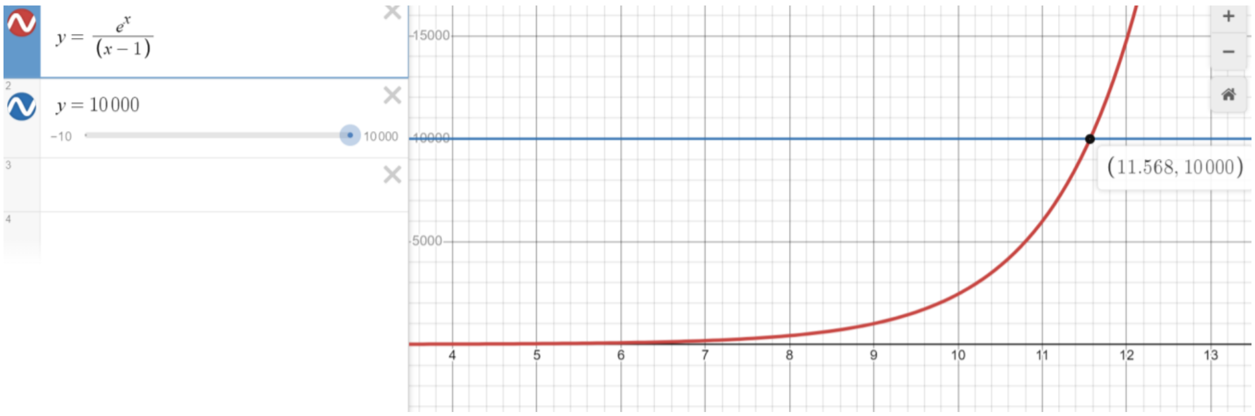
$$\sum ne^{-n}$$

How many terms do I need to get the error within $10^{-4} = 0.0001$?

$$\int_N^\infty xe^{-x}dx = \lim_{b\to\infty}\left(-be^{-b} - e^{-b}\right) - \left(-Ne^{-N} - e^{-N}\right) = Ne^{-N} - e^{-N} = 10^{-4}$$

$$e^{-N}(N-1) = 10^{-4}$$

$$10^4 = \frac{e^N}{N-1}$$



N=11.568… actually 12.
I need to add up 11 terms to get my sum with 0.0001 of the true value.

Series tests so far:
Two tests that allow us to determine the value a series converges to:
Geometric series test
Telescoping series test
Two tests that allow up to find an error on the true value of the sum:
Alternating Series Test (error = n+1st term)
Integral Test (error is integral from n+1 to infinity)
Our last test:
Test of Divergence

The P-series test

A p-series $a_n = \frac{1}{n^p}$, in the form $\sum \frac{1}{n^p}$

Use integral test to prove the p-series test. What values of p makes the p-series converge? And when diverge?

The integral test says that the series will converge if $\int_1^\infty \frac{1}{x^p}\,dx = \int_1^\infty x^{-p}\,dx$ converges, or the series will diverge if this integral diverges.

There is one exception to the power rule and that is when p=1.
Case 1: when p=1

$$\int_1^\infty \frac{1}{x}\,dx = \ln(x)|_1^\infty = \infty$$

Diverges

Case 2: when $p < 1$ (like ½?) clearly anything less than 0 will end up in the numerator and diverge.

$$\int_1^\infty x^{-p}\,dx = \frac{x^{-p+1}}{-p+1}\bigg|_1^\infty = \frac{1}{1-p}\left[x^{1-p}\right]_1^\infty =$$

Think about a specific example in this domain from (0 to 1)… p=1/2

$$\frac{1}{1-\frac{1}{2}}\left[x^{1-\frac{1}{2}}\right]_1^{\infty} = 2\left[x^{\frac{1}{2}}\right]_1^{\infty} = \infty$$

diverges

Case 3: when $p > 1$

$$\int_1^{\infty} x^{-p}\,dx = \frac{x^{-p+1}}{-p+1}\bigg|_1^{\infty} = \frac{1}{1-p}\left[x^{1-p}\right]_1^{\infty}$$

Specific case $p = 1.5$

$$\frac{1}{1-1.5}\left[x^{1-1.5}\right]_1^{\infty} = -2\left[x^{-\frac{1}{2}}\right]_1^{\infty} = -2[0-1] = 2$$

Converges

To sum up, a p-series $\sum \frac{1}{n^p}$ diverges if $p \leq 1$ and converges if $p > 1$.

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

Does the series converge? Therefore, the series does converge by the p-series test since p=2, p>1.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}}$$

Does the series converge? By the alternating series test, the series does converge because $\lim_{n\to\infty} \frac{1}{\sqrt[3]{n}} = 0.$
Does the series converge conditionally or absolutely?

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$$

This diverges by the p-series test, since $p \leq 1$ (p=1/3). Therefore, the series converges conditionally.

There is a similar test you can derive with the integral test for $\sum \frac{1}{n(\ln n)^p}$.

Comparison Tests
Direct Comparison and Limit Comparison Tests.

Direct Comparison

You are comparing two series. You want to test for convergence or divergence of $\sum a_n$, and you know the convergence or divergence of $\sum b_n$.

If b_n converges and if $a_n \leq b_n$ for all n, then a_n also converges.

If b_n diverges and if $a_n \geq b_n$ for all n, then a_n also diverges

Suppose $a_n = \frac{1}{(n+1)^2}$, and $b_n = \frac{1}{n^2}$. I know b_n converges by the p-series. Is $a_n \leq b_n$? Yes. Therefore, by the direct comparison test, $\sum a_n$ converges.

I could not use this test on $a_n = \frac{1}{n^2-1}$ because this denominator is smaller than n^2 and therefore the fraction would be larger.

If I was looking at divergence.

If $a_n = \frac{1}{n-\frac{1}{2}}$ and $b_n = \frac{1}{n}$. b_n diverges because of the p-series test. And $a_n \geq b_n$, so $\sum a_n$ also diverges

If you are comparing a convergent series:
You can add in the denominator or subtract in the numerator (to make the series smaller)

If you are comparing to a divergent series:
Then you can add in the numerator, or subtract in the denominator (to make the series bigger).

Limit Comparison
This test compares two series, one of which we know what it is doing.
We are testing a_n series, and we know whether b_n converges or diverges.

$$\lim_{n\to\infty} \frac{a_n}{b_n}$$

If the limit is a finite, non-zero value, then the two series converge or diverge together. If you get a 0 or ∞ then the comparison is a bad one and it can't tell you anything.

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{2n^3-1}}$$

$$\frac{1}{\sqrt{2n^3-1}} \cong \frac{1}{n^{\frac{3}{2}}}$$

$b_n = \frac{1}{n^{\frac{3}{2}}}$ converges by the p-series.

$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt{2n^3-1}}}{\frac{1}{n^{\frac{3}{2}}}} = \lim_{n\to\infty} \frac{n^{\frac{3}{2}}}{\sqrt{2n^3-1}} = \lim_{n\to\infty} \frac{n^{\frac{3}{2}} n^{-\frac{3}{2}}}{\sqrt{2n^3-1} \times n^{-\frac{3}{2}}} = \lim_{n\to\infty} \frac{1}{\sqrt{2-\frac{1}{n^3}}} = \frac{1}{\sqrt{2}}$$

The series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{2n^3-1}}$ converges by the limit comparison test.

We are up to 9 tests.

5 from earlier
Integral Test
P-series test
Direct comparison
Limit comparison

When you are doing series tests, try to think of these as “proofs”. For instance, if you use a comparison, you need to state what series you are comparing to, and what test you can use to determine convergence or divergence. And then also your conclusion of the comparison test.