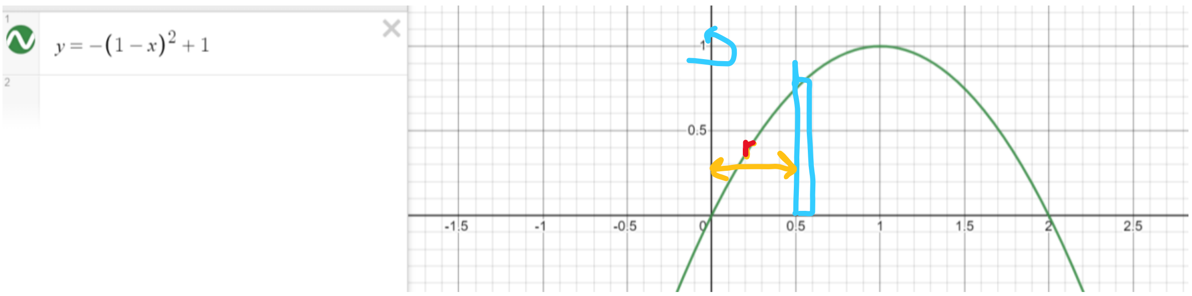
6/6/2022

Volumes of solids of Revolution—Cylindrical Shells (2.3)
Arc Length, and Surface Area of solids of revolution (2.4)

The method of cylindrical shells allows us to rotate functions of x around the y-axis (or lines parallel to the y-axis). Or to rotate functions of y around the x-axis (or a line parallel to x-axis).



Surface area times the thickness is the volume. And the surface of the cylindrical shell is a rectangle. The height of the cylinder = function. The other side is the circumference of the cylinder. $C = 2\pi r$. What is the radius? That distance is just x. The volume is going to be $V = (2\pi x)f(x)\Delta x$.

$$V = 2\pi \int_a^b x f(x) dx$$

$$V = 2\pi \int_c^d y f(y) dy$$

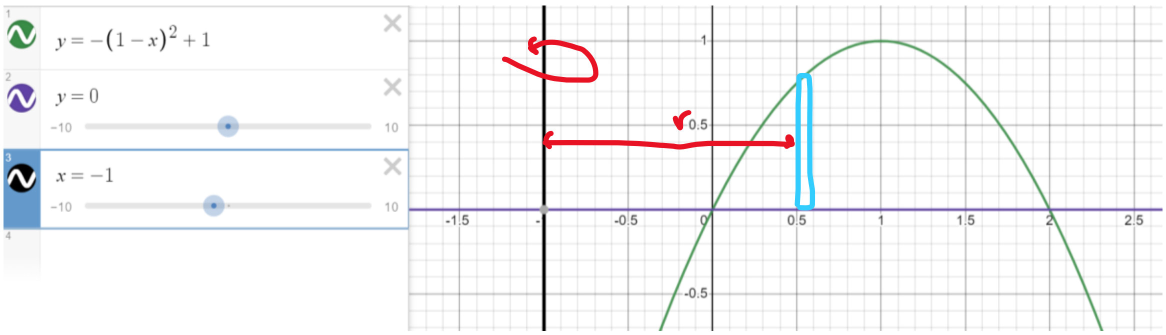
Example.
Find the volume of the solid of revolution bounded by $f(x) = -(1-x)^2 + 1$ and $y = 0$. Revolved around the y-axis.

$$f(x) = -(1-x)^2 + 1 = -(1 - 2x + x^2) + 1 = -1 + 2x - x^2 + 1 = 2x - x^2$$

$$V = 2\pi \int_0^2 x(2x - x^2)dx = 2\pi \int_0^2 2x^2 - x^3 dx = 2\pi \left[\frac{2}{3}x^3 - \frac{1}{4}x^4\right]_0^2 =$$

$$2\pi \left[\frac{16}{3} - 4\right] = 2\pi \left[\frac{4}{3}\right] = \frac{8\pi}{3}$$

What if I'm rotating the function around a line parallel to the y-axis?

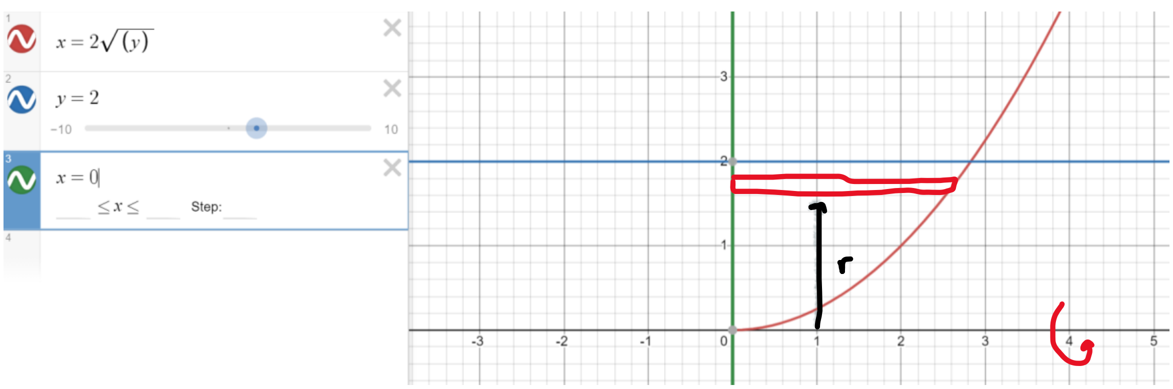


What changes here is the radius in our function. It's not x anymore. It's now x minus the axis of rotation.

$$x-(-1)=x+1$$

Repeat the same problem as before with the new axis of rotation.

$$V=2\pi\int_0^2 (x+1)(2x-x^2)dx$$

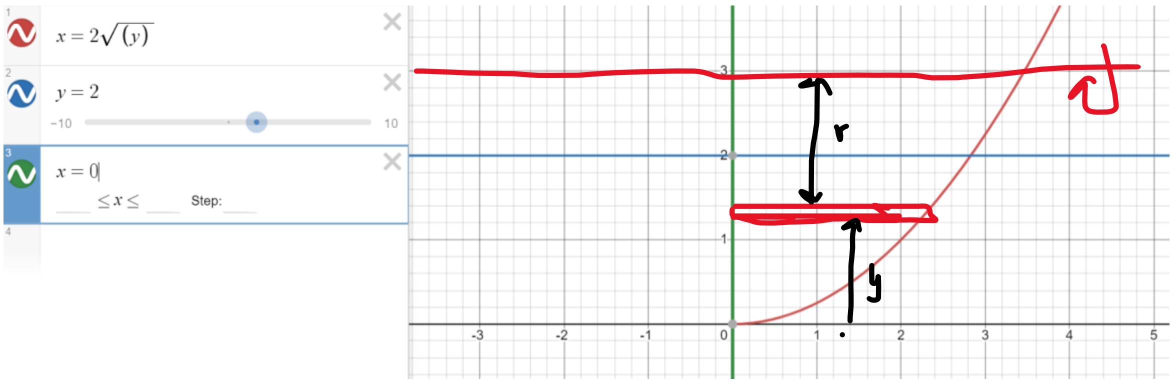


Example.

Find the volume of the solid of revolution if we rotate the region bounded by $x=2\sqrt{y}, x=0, y=2$ around the x-axis.

$$V=2\pi\int_c^d yf(y)dy=2\pi\int_0^2 y(2\sqrt{y})dy=4\pi\int_0^2 y^{\frac{3}{2}}dy=4\pi\left[\frac{2}{5}y^{\frac{5}{2}}\right]_0^2=\frac{8\pi}{5}(\sqrt{32})=\frac{8\pi}{5}(4\sqrt{2})$$

$$\frac{32\pi\sqrt{2}}{5}$$



Rotate the same problem but around y=3.

In this orientation the axis of rotation is on the opposite side of the region (compared to the x-axis). The radius for the cylinder, becomes axis of rotation minus y.

$$2\pi\int_0^2 (3-y)\left(2\sqrt{y}\right)dy$$

Arc Length and Surface Area of volumes of revolution.

The approximation that underlies the arc length formula is the Pythagorean Theorem.

$$a^2+b^2=c^2$$

a is the small distance in x = Δx, and b is the small distance in y = Δy, and c is the estimate for the length of the curve.

See handout on Arc Length for details of the derivation.

$$s=\int_a^b \sqrt{1+[f'(x)]^2}dx$$

Example. Find the arclength of the function $f(x)=\ln(\cos x)$ on the interval $\left[\frac{\pi}{6},\frac{\pi}{3}\right]$.

$$f'(x)=\frac{1}{\cos x}(-\sin x)=-\tan x$$

$$s=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\sqrt{1+(-\tan x)^2}dx=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\sqrt{1+\tan^2 x}\,dx=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\sqrt{\sec^2 x}\,dx=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\sec x\,dx=$$

$$[\ln|\sec x+\tan x|]_{\frac{\pi}{6}}^{\frac{\pi}{3}}=\ln\left|2+\sqrt{3}\right|-\ln\left|\frac{2}{\sqrt{3}}+\frac{1}{\sqrt{3}}\right|\approx 0.76765\ldots$$

Very few functions make integrable arc length problems.

This leads us to the surface are of the solid of revolution.

$$SA = 2\pi \int_a^b R(x)\sqrt{1 + [f'(x)]^2}dx$$

The R(x) function out front is going to change depending on the axis of rotation.

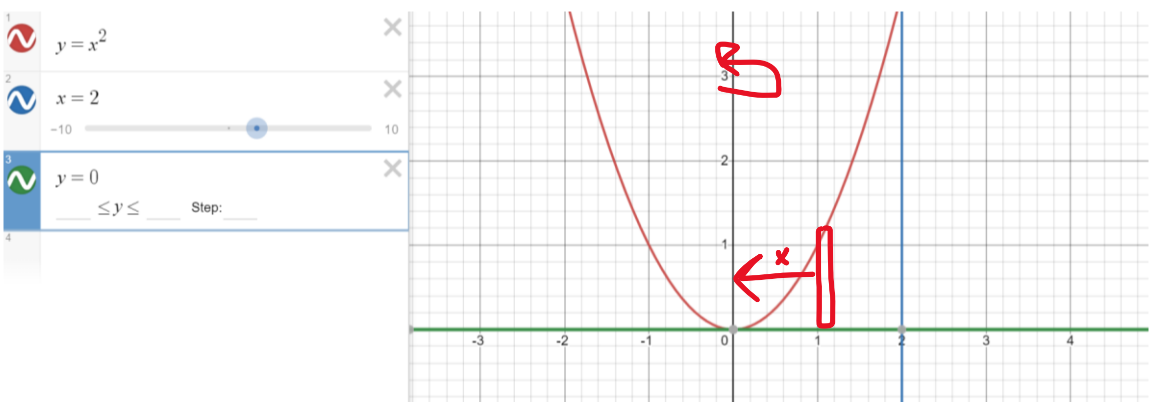
If we are using a function of x, and we are rotating around the x-axis, the radius of cylinders is the height of the function. So we'd get

$$SA = 2\pi \int_a^b f(x)\sqrt{1 + [f'(x)]^2}dx$$

If we are using a function of x, and we are rotating around the y-axis, the radius of our cylinders is the distance from y-axis, which is x. So we get

$$SA = 2\pi \int_a^b x\sqrt{1 + [f'(x)]^2}dx$$

Example.
We want to find the surface of the solid of revolution formed by rotating the region bounded by $y = x^2$ and $x = 2$, $y = 0$, around the y-axis.



$$SA = 2\pi \int_a^b x\sqrt{1 + [f'(x)]^2}dx$$

$$f'(x) = 2x$$

$$2\pi \int_0^2 x\sqrt{1 + [2x]^2}dx = 2\pi \int_0^2 x\sqrt{1 + 4x^2}dx$$

$$u = 1 + 4x^2, du = 8xdx, \frac{1}{8}du = xdx$$

$$2\pi \int_1^{17} \frac{1}{8}u^{1/2}du = \frac{\pi}{4}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_1^{17} = \frac{\pi}{6}\left[17^{\frac{3}{2}} - 1\right]$$

What if we rotate around the x-axis?

$$SA = 2\pi \int_a^b f(x)\sqrt{1 + [f'(x)]^2}dx = 2\pi \int_0^2 x^2\sqrt{1 + [2x]^2}dx = 2\pi \int_0^2 x^2\sqrt{1 + 4x^2}dx$$

This would require trig substitution to complete by hand (next chapter), or we can put it in the calculator.