6/27/2022

Integration by Tables (3.5)
Numerical Integration (3.6)
Improper Integrals (3.7)
Review for Exam

Integration by Tables
In this course, integration by tables should be an option of last resort unless you are specifically asked to use it.

Example.

$$\int \frac{\sqrt{x^2-9}}{3x}\,dx = \frac{1}{3}\int \frac{\sqrt{x^2-9}}{x}\,dx$$

$$a = 3$$

$$\int \frac{\sqrt{x^2-a^2}}{x}dx = \sqrt{x^2-a^2} - a\,\text{arcsec}\left|\frac{x}{a}\right| + C$$

3.32 formula

$$= \frac{1}{3}\left[\sqrt{x^2-9} - 3\,\text{arcsec}\left|\frac{x}{3}\right|\right] + C = \frac{\sqrt{x^2-9}}{3} - \text{arcsec}\left|\frac{x}{3}\right| + C$$

Example.

$$\int \frac{\cos x}{\sin^2 x + 2\sin x}\,dx = \int \frac{\cos x}{\sin x\,(\sin x + 2)}\,dx$$

$$u = \sin x\,,\, du = \cos x\,dx$$

$$\int \frac{1}{u(u+2)}\,du = \int \frac{1}{u^2+2u}\,du$$

$$\int \frac{1}{x(ax+b)}\,dx = \frac{1}{b}\ln\left|\frac{x}{ax+b}\right| + C$$

formula 2.17

$$a = 1, b = 2$$

$$\frac{1}{2}\ln\left|\frac{u}{u+2}\right| + C = \frac{1}{2}\ln\left|\frac{\sin x}{\sin x + 2}\right| + C$$

Numerical Integration Techniques

Trapezoidal Rule, and Simpson's Rule.

Trapezoidal Rule

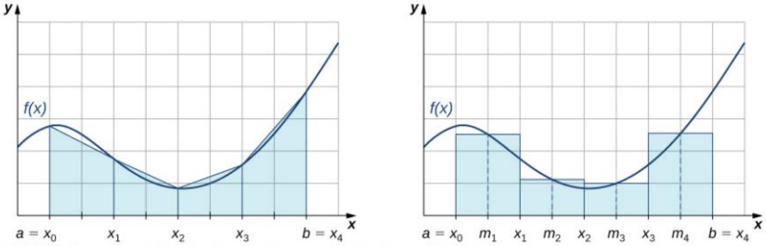

Figure 3.15 The trapezoidal rule tends to be less accurate than the midpoint rule.

$$A_{trapezoid} = \frac{1}{2}(b_1 + b_2)h = \frac{1}{2}\big(f(x_i) + f(x_{i+1})\big)\Delta x$$

$$\Delta x = \frac{b-a}{n}$$

$$\frac{\frac{1}{2}\big(f(x_i) + f(x_{i+1})\big)(\mathrm{b}-\mathrm{a})}{\mathrm{n}} = \frac{b-a}{2n}[f(x_i) + f(x_{i+1})]$$

Add up all the trapezoids

$$\int_a^b f(x)dx \approx \frac{b-a}{2n}[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

Example.

$$\int_1^3 x^3\,dx$$

Estimate it with the trapezoidal rule, with n=6.

$$b-a = 2, \frac{b-a}{n} = \frac{2}{6} = \frac{1}{3}$$

Points to be evaluated $\left\{1, \frac{4}{3}, \frac{5}{3}, 2, \frac{7}{3}, \frac{8}{3}, 3\right\}$

$$\int_1^3 x^3\,dx \approx \frac{2}{12}\left[f(1) + 2f\left(\frac{4}{3}\right) + 2f\left(\frac{5}{3}\right) + 2f(2) + 2f\left(\frac{7}{3}\right) + 2f\left(\frac{8}{3}\right) + f(3)\right] =$$

$$\frac{1}{6}\left[1^3 + 2\left(\frac{4}{3}\right)^3 + 2\left(\frac{5}{3}\right)^3 + 2(2)^3 + 2\left(\frac{7}{3}\right)^3 + 2\left(\frac{8}{3}\right)^3 + (3)^3\right] = 20.22222\ldots$$

True value of the integral is 20.

Error formula for trapezoidal rule

$$T_n \leq \frac{M(b-a)^3}{12n^2} = \frac{\left|\max\left(f''(x)\right)\right|(b-a)^3}{12n^2}$$

M is the maximum value of $f''(x)$ on the interval.

$$f(x) = x^3, f'(x) = 3x^2, f''(x) = 6x$$

$$\max\left(f''(x)\right)_{on\ [1,3]} = 6(3) = 18$$

$$Error \leq \frac{\left|\max\left(f''(x)\right)\right|(b-a)^3}{12n^2} = \frac{(18)(2)^3}{12(6)^2} = 0.333\ldots$$

To find the value of n needed to obtain a given error:

$$n^2 \geq \frac{\left|\max\left(f''(x)\right)\right|(b-a)^3}{12(Error)}$$

The value of n here is a minimum n required to obtain the specified error. If you get a decimal, you must round up to the next whole integer.

Error to be less than 10^{-4}?

$$n^2 \geq \frac{\left|\max\left(f''(x)\right)\right|(b-a)^3}{12(Error)} = \frac{(18)(2)^3}{12 \times 10^{-4}} = 120{,}000$$

$$n \geq 346.41\ldots, n = 347$$

Simpson's Rule

Uses a quadratic function to estimate the curve between three points on the function. I won't derive the formula here because it requires a lemma to obtain the quadratic approximation rule.

The number of intervals has to be even for Simpson's rule.

Formula we obtain:

$$\int_a^b f(x)dx \approx$$

$$\frac{b-a}{3n}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$

$$\int_1^3 x^3\,dx$$

Estimate it with the Simpson's rule, with n=6.

$$b - a = 2, \frac{b-a}{n} = \frac{2}{6} = \frac{1}{3}$$

Points to be evaluated $\left\{1, \frac{4}{3}, \frac{5}{3}, 2, \frac{7}{3}, \frac{8}{3}, 3\right\}$

$$\int_1^3 x^3\, dx \approx$$

$$\frac{2}{3(6)}\left[f(1) + 4f\left(\frac{4}{3}\right) + 2f\left(\frac{5}{3}\right) + 4f(2) + 2f\left(\frac{7}{3}\right) + 4f\left(\frac{8}{3}\right) + f(3)\right] =$$

$$\frac{1}{9}\left[(1)^3 + 4\left(\frac{4}{3}\right)^3 + 2\left(\frac{5}{3}\right)^3 + 4(2)^3 + 2\left(\frac{7}{3}\right)^3 + 4\left(\frac{8}{3}\right)^3 + (3)^3\right] = 20$$

Error formula

$$S_n \leq \frac{|\max(f^{IV}(x))|(b-a)^5}{180n^4} = \frac{\left|\max\left(f^{(4)}(x)\right)\right|(b-a)^5}{180n^4}$$

$$f''(x) = 6x, f'''(x) = 6, f^{IV}(x) = 0$$

Rearrange the formula for n

$$n^4 \geq \frac{|\max(f^{IV}(x))|(b-a)^5}{180(Error)}$$

Suppose we want to estimate $\int_1^4 \ln(x) + 1 dx$ with Simpson's rule, to an error of less than 10^{-4}

$$f(x) = \ln(x) + 1$$

$$f'(x) = \frac{1}{x} = x^{-1}$$

$$f''(x) = -x^{-2}$$

$$f''(x) = 2x^{-3}$$

$$f^{IV}(x) = -6x^{-3} = -\frac{6}{x^3}$$

$$n^4 \geq \frac{\left|-\frac{6}{1^3}\right|(3)^5}{180(10^{-4})} = 81{,}000$$

$$n \geq 16.870 \ldots n = 18$$

(n must be even!)

Simpson's rule is what your calculator is doing.

Improper Integrals

These are integrals that have problems.

They are undefined at an endpoint, or they are undefined at some point in the interval of integration, or one (or both) limits are at infinity.

The general strategy is 1) identify the problem points (either of the limits or inside the interval).

If the point is inside the interval, split the integral at the problem point and then follow the steps for each integral separately.
If both limits are a problem, split the integral into two pieces (in between the two limits), and then follow the steps for one bad limit with each integral.

2) replace the bad limit with a dummy constant. Use a for the lower limit and b for the upper limit. Some books use r for either.

3) Then integrate the function using the dummy limit (fundamental theorem of calculus)

4) take the limit as the dummy variable goes to the value of the problem limit.

If you get a finite value, that's the value of the integral and we say the integral converges.
If the limit goes to infinity (split integrals must be treated separately), then we say the integral diverges.

Look at an example.

$$\int_{-1}^{1} \frac{1}{\sqrt[3]{x}} dx$$

Problem here is that $x = 0$ is not defined.

$$\int_{-1}^{0} \frac{1}{\sqrt[3]{x}} dx + \int_{0}^{1} \frac{1}{\sqrt[3]{x}} dx = \lim_{b\to 0} \int_{-1}^{b} \frac{1}{\sqrt[3]{x}} dx + \lim_{a\to 0} \int_{a}^{1} \frac{1}{\sqrt[3]{x}} dx =$$

$$\lim_{b\to 0} \int_{-1}^{b} x^{-\frac{1}{3}} dx + \lim_{a\to 0} \int_{a}^{1} x^{-\frac{1}{3}} dx = \lim_{b\to 0} \frac{3}{2} x^{\frac{2}{3}} \Big|_{-1}^{b} + \lim_{a\to 0} \frac{3}{2} x^{\frac{2}{3}} \Big|_{a}^{1}$$

$$= \lim_{b\to 0} \left(\frac{3}{2} b^{\frac{2}{3}} - \frac{3}{2}(-1)^{\frac{2}{3}}\right) + \lim_{a\to 0} \left(\frac{3}{2}(1)^{\frac{2}{3}} - \frac{3}{2} a^{\frac{2}{3}}\right) = \lim_{b\to 0} \left(0 - \frac{3}{2}(-1)^{\frac{2}{3}}\right) + \lim_{a\to 0} \left(\frac{3}{2}(1)^{\frac{2}{3}} - 0\right) = 0$$

We can't add the two pieces back together unless both halves converge.

In some problems, depending on the function, you may need to apply L'Hopital's rule.

Exam #1
The exam covers basically Ch 2 and Ch 3 from online textbook.

- Area between 2 curves
- Solids of revolution
- Arc length
- Surface area for solids of revolution
- Work
- The rest of chapter 2 was mostly "review", except hyperbolic trig functions
- Integration by parts
- Trig function integrals
- Trig substitution

- Partial fractions
- Integration by tables is fair game
- Numerical integration
- Improper Integral problem