6/1/2022

Calc II:
Area between Curves
Solids of Revolution: Washer/Disk Method

Area between curves.
When we integrated a single function, it was the area between that function and the axis (typically, the area between f(x) and the x-axis, which is y=0.

$$\int_a^b [f(x) - 0]dx = \int_a^b f(x)dx$$

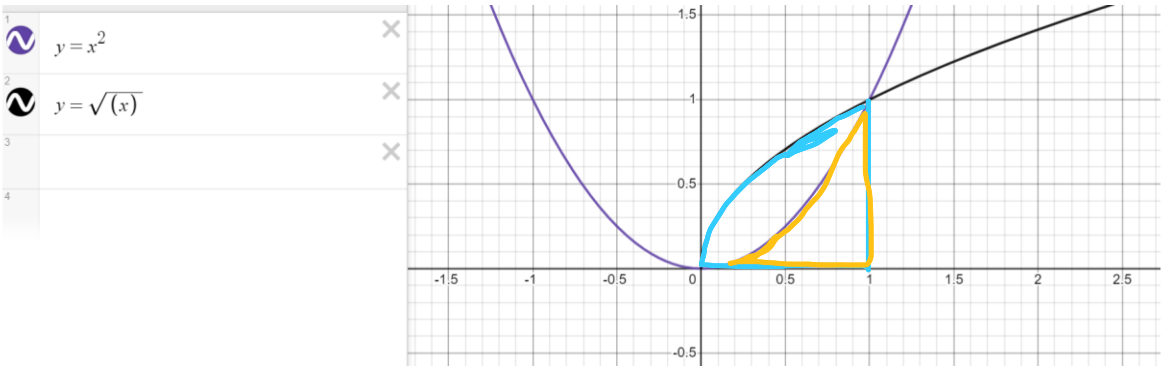
Height is the difference between the top function height and the bottom function height.

$$(4 - x^2) - (x^2 - 4)$$

In our general case, the area between two curves is

$$\int_a^b f(x)_{top} - g(x)_{bottom} dx$$

Another way to think about this.



What is the area under the square root curve?
Everything inside this blue box. $\int_0^1 \sqrt{x}dx$

What part of this don't we need for the area between the curves?
It's the area under the x^2 curve. So the area under that curve is $\int_0^1 x^2\,dx$, and we subtract that from the area under the square root curve, and we left with just the area between them.

$$A = \int_0^1 \sqrt{x}dx - \int_0^1 x^2\,dx = \int_0^1 \sqrt{x} - x^2 dx = \int_a^b f(x) - g(x)dx$$

Solution

The region is depicted in the following figure.

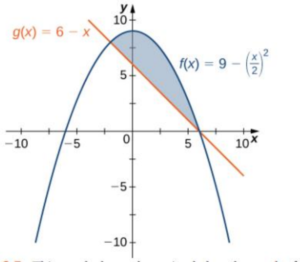


Figure 2.5 This graph shows the region below the graph of $f(x)$ and above the graph of $g(x)$.

Find the area between the curves $y = 6 - x, y = 9 - \frac{x^2}{4}$.

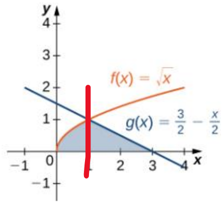
$$6 - x = 9 - \frac{x^2}{4}$$
$$-3 - x = -\frac{x^2}{4}$$
$$12 + 4x = x^2$$
$$x^2 - 4x - 12 = 0$$
$$(x - 6)(x + 2) = 0$$
$$x = -2,6$$

$$\int_{-2}^{6} 9 - \frac{x^2}{4} - (6 - x)dx = \int_{-2}^{6} 3 + x - \frac{x^2}{4} dx = 3x + \frac{1}{2}x^2 - \frac{x^3}{12}\Big|_{-2}^{6} =$$

$$18 + 18 - 18 - \left(-6 + 2 + \frac{2}{3}\right) = 18 - \left(-\frac{10}{3}\right) = \frac{64}{3}$$

Let's look at an example with functions of y: $f(y)$.

2.5 Let's revisit the checkpoint associated with **Example 2.4**, only this time, let's integrate with respect to y. Let be the region depicted in the following figure. Find the area of R by integrating with respect to y.



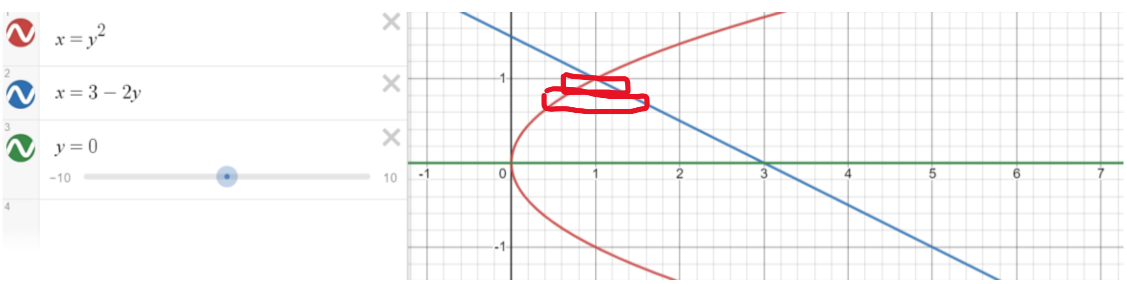
Find the area bounded by $y = \sqrt{x}, y = \frac{3}{2} - \frac{x}{2}, y = 0$

In this orientation: the area is $\int_0^1 \sqrt{x}dx + \int_1^3 \frac{3}{2} - \frac{x}{2}dx$

Solve for x to switch to the horizontal orientation.

$$y = \sqrt{x} \rightarrow x = y^2$$

$$y = \frac{3}{2} - \frac{x}{2} \rightarrow 2y = 3 - x \rightarrow x = 3 - 2y$$



Top function = rightmost function

Bottom function = leftmost function

$$\int_c^d f(y)_{right} - g(y)_{left}\,dy$$

$$\int_0^1 (3-2y) - (y^2)dy = \int_0^1 3 - 2y - y^2 dy = 3y - y^2 - \frac{1}{3}y^3\Big|_0^1 = 3 - 1 - \frac{1}{3} = \frac{5}{3}$$

Solids of Revolution (2.2)

Disk Method/Washer Method

The general idea is to estimate a formula using rectangles and then make them smaller until the limit.

Top of the rectangle is the height of the function = height of the circle at that point. The width of the rectangle is Δx.

When you rotate this rectangle around the x-axis, you get a cylinder, and the radius of that cylinder is the height of the rectangle. What is the volume of the cylinder?

In general : $V = \pi r^2 h$

In this case: $\pi\left(height\ of\ the\ circle = f(x)\right)^2 \Delta x$

So the volume is going to be the sum of our cylinders

$$\lim_{n\to\infty} \sum_{i=1}^{n} \pi[f(x)]^2 \Delta x = \int_a^b \pi[f(x)]^2 dx = V$$

Disk method where the bottom of area under the function touches the x-axis (axis of rotation).

The washer method, except there is a top function and bottom function which does not touch the axis of rotation everywhere.

$$V = \int_a^b \pi\{[f(x)]^2 - [g(x)]^2\} dx = \pi \int_a^b R_{outer}^2 - R_{inner}^2 dx$$

If your function is a function of x, and you are rotating around the x-axis (or any line parallel to the x-axis) use the washer/disk method.

If your function is a function of y, and you are rotating around the y-axis (or any line parallel to the y-axis), use the washer/disk method.

To continue rotating our half circle.

$$Outer\ Radius = \sqrt{16 - x^2}, Inner\ Radius = 0$$

$$V = \pi \int_{-4}^{4} \left(\sqrt{16 - x^2}\right)^2 dx = \pi \int_{-4}^{4} 16 - x^2 dx = 2\pi \int_0^4 16 - x^2 dx = 2\pi[16x - \frac{1}{3}x^3\Big|_0^4 =$$

$$2\pi\left[64 - \frac{64}{3}\right] = 2\pi\left[\frac{128}{3}\right] = \frac{256}{3}\pi$$

What is the volume of the sphere with a radius of 4?

$$V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(4)^3 = \frac{256}{3}\pi$$

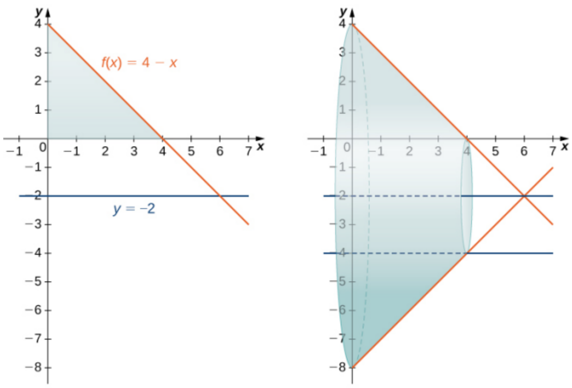
Example 2.11

The Washer Method with a Different Axis of Revolution

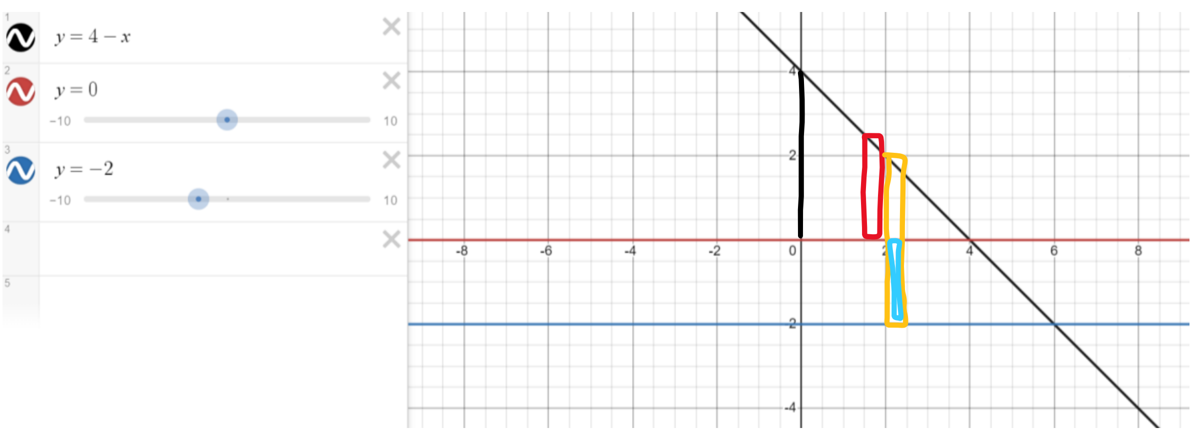
Find the volume of a solid of revolution formed by revolving the region bounded above by $f(x) = 4 - x$ and below by the x-axis over the interval $[0, 4]$ around the line $y = -2$.

Solution

The graph of the region and the solid of revolution are shown in the following figure.



What is changing when we rotate around a line parallel to the x-axis (a y=constant line). What changes is not the formula, the function of the inner and outer radius changes.



That orange distance is the height of the function minus the axis of rotation.

$$4 - x - (-2) = 4 - x + 2$$

The blue distance is the height of the inner function minus the axis of rotation

$$0 - (-2) = 2$$

Outer radius = $4 - x + 2 = 6 - x$

Inner radius = 2

$$V = \pi\int_0^4 (6-x)^2 - (2)^2 dx = \pi\int_0^4 32 - 12x + x^2 dx = 32x - 6x^2 + \frac{1}{3}x^3\Big|_0^4 = \frac{160\pi}{3}$$

$$128 - 96 + \frac{64}{3} = \frac{160}{3}$$

Geometric area is positive. If you are getting a negative area, you've made a mistake. The functions might cross a third time, so check for additional intersections. Or you subtracted in the wrong order.

Likewise for volume, it must be positive. But it could be due to another error so check for sign errors.

The derivative rule is $\frac{d}{dx}[x^n] = nx^{n-1}$

Antiderivative rule: $\int x^n dx = \frac{x^{n+1}}{n+1}$

$$\int x^2 dx = \frac{x^3}{3}$$

$$\frac{d}{dx}\left[\frac{x^3}{3}\right] = \frac{1}{3}\frac{d}{dx}[x^3] = \frac{1}{3}(3x^2) = x^2$$