MTH 264, Exam #1, Summer 2022 Name KEY

Instructions: Work the problems below as directed. Show all work. Clearly mark your final answers. Use exact values unless the problem specifically directs you to round. Simplify as much as possible. Partial credit is possible, but solutions without work will not receive full credit.

Part 1: These questions you will submit answers to in Canvas. Show all work and submit the work with Part 2 of the exam. But you must submit the answers in Canvas to receive credit. Each question/answer will be listed separately. The Canvas question will refer to the number/part to indicate where you should submit which answer. The questions will appear in order (in case there is an inadvertent typo). Correct answers will receive full credit with or without work in this section, but if you don't submit work and clearly label your answers, you won't be able to challenge any scoring decisions for making an error of any kind.

1. Find the area between the curves $f(x) = -x^2 + 4x + 2, g(x) = x + 2$. (10 points)

$-x^2 + 4x + \cancel{2} = x + \cancel{2}$
$\quad -x \qquad\quad -x$

$-x^2 + 3x = 0$

$-x(x-3) = 0$

$x = 0,\ x = 3$

$\int_0^3 (-x^2+4x+2) - (x+2)\,dx =$

$\int_0^3 -x^2 + 3x\,dx =$

$-\frac{1}{3}x^3 + \frac{3}{2}x^2 \Big|_0^3 = -\frac{1}{3}(27) + \frac{3}{2}(9) =$

$-9 + \frac{27}{2} = \boxed{\frac{9}{2}} = 4.5$

2. Find the volume of the solid of revolution bounded by the graphs of $y = \sqrt{16 - x^2}$, $x = 0$ revolved around the x-axis. (10 points)

$\pi\int_0^4 (\sqrt{16-x^2})^2\,dx = \pi\int_0^4 16 - x^2\,dx = \pi\left[16x - \frac{1}{3}x^3\right|_0^4$

$= \pi\left[64 - \frac{64}{3}\right] = \boxed{\frac{128\pi}{3}} \approx 134.041$

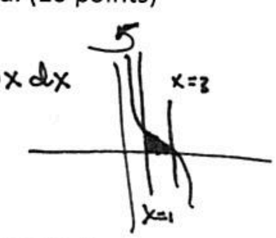
$(x-2)^3 = x^3 - 6x^2 + 12x - 8 \qquad -(x-2)^3 + 2 = -x^3 + 6x^2 - 12x + 8 + 2 = -x^3 + 6x^2 - 12x + 10$

3. Find the volume of the solid of revolution bounded by the graphs of $y = -(x-2)^3 + 2$, $y = 0$, $x = 1$ and $x = 3$, revolved around the y-axis using the shell method. (10 points)

$2\pi\int_1^3 x\left(-(x-2)^3 + 2\right)dx = 2\pi\int_1^3 -x^4 + 6x^3 - 12x^2 + 10x\,dx$

$= 2\pi\left[-\frac{x^5}{5} + \frac{6}{4}x^4 - \frac{12}{3}x^3 + \frac{10}{2}x^2\right|_1^3 = 2\pi\left(\frac{38}{5}\right) =$

$\boxed{\frac{76\pi}{5}} \approx 47.7522$



4. Find the length of the arc of the curve $f(x) = \cosh x$ on the interval [0,2]. Round to 4 decimal places. (9 points)

$y' = \sinh x$ $\cosh^2 x - \sinh^2 x = 1$

$\int_0^2 \sqrt{1+\sinh^2 x}\, dx = \int_0^2 \sqrt{\cosh^2 x}\, dx = \int_0^2 \cosh x\, dx = \sinh x \Big|_0^2$

$\boxed{\sinh(2)} \approx 3.62686$

5. Find the surface area of the shape formed by revolving the graph $y = x^3$ around the y-axis, on the interval [0,2]. Sketch the region. (10 points)

$2\pi \int_0^2 x\sqrt{1+(3x^2)^2}\, dx = 2\pi \int_0^2 x\sqrt{1+9x^4}\, dx$

$\frac{2\pi}{12}\left[\ln\left(\sqrt{1+9x^4}+3x^2\right) + 3x^2\sqrt{9x^4+1}\,\right]_0^2 =$ $y' = 3x^2$

$\pi/6\left[\ln\left(\sqrt{145}+12\right) + 12\sqrt{145}\right] \approx 12.3066$

6. Find the center of mass of the lamina with constant density of the region bounded by $y = x^2 - x, y = 0$. (16 points)

$M = \int_0^1 x^2 - x\, dx = \frac{1}{3}x^3 - \frac{1}{2}x^2 \Big|_0^1 = \frac{-1}{6}$ $\boxed{\left(\frac{1}{2}, -\frac{1}{10}\right)}$

$M_x = \frac{1}{2}\int_0^1 (x^2-x)^2\, dx = \frac{1}{2}\int_0^1 x^4 - 2x^3 + x^2\, dx = \frac{1}{2}\left[\frac{1}{5}x^5 - \frac{2}{4}x^4 + \frac{1}{3}x^3\right]_0^1 = \frac{1}{60}$

$M_y = \int_0^1 x(x^2-x)\, dx = \int_0^1 x^3 - x^2\, dx = \frac{1}{4}x^4 - \frac{1}{3}x^3 \Big|_0^1 = -1/12$ $\bar{x} = \frac{M_y}{M} = \frac{-1/12}{-1/6} = \frac{1}{2}$

$\bar{y} = M_x/M = \frac{1}{60} / -\frac{1}{6} = \frac{-1}{10}$

7. A force of 5 pounds compresses a 15-inch spring a total of 4 inches. How much work is done in compressing the spring 7 inches? (10 points) in foot-pounds

$F = kx$ $5 = k\frac{4}{12} \rightarrow k = \frac{5 \cdot \cancel{12}^{3}}{\cancel{4}} = 15$ $F = 15x$

$\int_0^{7/12} 15x\, dx = \frac{15}{2}x^2 \Big|_0^{7/12} = \frac{15}{2}\left(\frac{7}{12}\right)^2 = \frac{245}{96} \approx 2.55208$ ft-pounds

8. For the following integrals, state which method you would use, and which basic integration rule. Do not actually perform the integration. Methods may include: substitution, change of variables, complete the square, add/subtract, trig identities, long division, partial fractions, by parts, trig substitution, etc. Basic integration rules may include: power rule, log rule, exponential rule, trig functions, inverse trig functions, etc. Some problems may require more than more method or rule. (6 points each)

a. $\int \frac{3}{2-e^x}dx$ algebra, u-sub w/ e^{-x} log rule

b. $\int \cot x dx$ u-sub ($u = \sin x$) log rule

c. $\int x^2 e^{x^2} dx$ by parts $u = x$, $dv = xe^{x^2}$

d. $\int \frac{1}{\sqrt{x^2-7}}dx$ trig sub

e. $\int \frac{1}{\sqrt{x}\sqrt{7-x}}dx$ trig sub

f. $\int \cos^2 x dx$ algebra trig functions

9. Use Simpson's Rule to approximate the area under the curve of $\int_1^2 e^x \ln x dx$ for n=6. (10 points)

$$\approx \frac{b-a}{3n}\left[f(x_0) + \cancel{2}4f(x_1) + 2f(x_2) + \cdots + 4f(x_{n-1}) + f(x_n)\right] = \frac{2-1}{3(6)}\left[e^1\ln(1) + 4e^{7/6}\ln(7/6) + 2e^{8/6}\ln(8/6)\right.$$
$$\left. + 4e^{9/6}\ln(9/6) + 2e^{10/6}\ln(10/6) + 4e^{11/6}\ln(11/6) + e^2\ln 2\right] =$$

$$2.06261$$

Part 2: In this section you will record your answers on paper along with your work. After scanning, submit them to a Canvas dropbox as directed. These questions will be graded by hand.

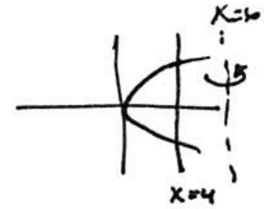
10. Use the definition of the hyperbolic trig functions to prove that the derivative of $\tanh(x) = \operatorname{sech}^2(x)$. (15 points)

$\tanh x = \frac{\sinh x}{\cosh x}$ use quotient rule

$$\left(\frac{\sinh x}{\cosh x}\right)' = \frac{\cosh x \cdot \cosh x - \sinh x \cdot \sinh x}{\cosh^2 x} = \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x$$

11. Find the volume of the solid generated by revolving the region bounded by the graphs of the equations $f(y) = y^2$ and $g(y) = 4$ about the line $x = 6$. Sketch the graph of the region. (15 points)

$$\pi\int_{-2}^{2}(6-y^2)^2-(6-4)^2\,dy = 2\pi\int_{0}^{2}36-12y^2+y^4-4\,dy =$$



$$2\pi\int_{0}^{2}32-12y^2+y^4\,dy = 2\pi\left[32y-4y^3+\frac{1}{5}y^5\right|_{0}^{2} =$$

$$2\pi\left(\frac{192}{5}\right) = \frac{384\pi}{5}$$

12. Consider a conical tank with diameter 14 feet and a height of 12 feet. Find the amount of work needed to drain the tank if it is only half full. Assume that the fluid in the tank is water and it has a weight-density of 62.4 lbs./ft³. (20 points)

$$\int_{0}^{6}62.4\cdot\frac{49\pi}{144}y^2(12-y)\,dy =$$

$$62.4\cdot\frac{49\pi}{144}\int_{0}^{6}12y^2-y^3\,dy = 62.4\cdot\frac{49\pi}{144}\left[4y^3-\frac{1}{4}y^4\right|_{0}^{6} =$$

$$62.4\cdot\frac{49\pi}{144}[540] = 11466\pi$$

$\approx 36{,}021.5$ ft-pounds

$\frac{7}{12}=\frac{r}{h} \to r=\frac{7}{12}h$

$A=\pi r^2=\frac{49}{144}\pi y^2$

13. Set up (but do not solve) this rational expression $\dfrac{x^3+5x^2-8x-24}{(x^2+1)^2(x-3)^3(x+4)(x-1)}$ for decomposition by partial fractions. (15 points)

$$\frac{Ax+B}{x^2+1}+\frac{Cx+D}{(x^2+1)^2}+\frac{E}{x-3}+\frac{F}{(x-3)^2}+\frac{G}{(x-3)^3}+\frac{H}{x+4}+\frac{I}{x-1}$$

14. Integrate by an appropriate method. (10 points each)

a. $\int \sin 2\theta \cos 2\theta d\theta$ $= \frac{1}{2}\int \sin 4\theta \, d\theta$ $= \boxed{-\frac{1}{8}\cos 4\theta + C}$

or $u = \sin 2\theta$, $du = 2\cos 2\theta \, d\theta$ (or switch)

$\frac{1}{2}\int u \, du = \frac{1}{4}u^2 + C = \boxed{\frac{1}{2}\sin^2 2\theta + C}$ or see back page

b. $\int x\sqrt{2x+3}dx$ $u = x$, $dv = (2x+3)^{1/2}dx$ for change of variables method, see back page

$du = dx$ $v = \frac{2}{3}(2x+3)^{3/2} \cdot \frac{1}{2}$

$\frac{1}{3}x(2x+3)^{3/2} - \int \frac{1}{3}(2x+3)^{3/2}dx$

$\frac{1}{3}x(2x+3)^{3/2} - \frac{1}{3}\cdot\frac{2}{5}(2x+3)^{5/2}\cdot\frac{1}{2} + C = \boxed{\frac{1}{3}x(2x+3)^{3/2} - \frac{1}{5}(2x+3)^{5/2} + C}$

c. $\int \sec^3 x \tan x dx$ $\int \sec^2 x (\sec x \tan x) dx$ $u = \sec x$

$\int u^2 du = \frac{1}{3}u^3 + C$ $du = \sec x \tan x dx$

$\boxed{\frac{1}{3}\sec^3 x + C}$

d. $\int \frac{3x-4}{(x-5)(x+1)}dx$ $\frac{A}{x-5} + \frac{B}{x+1} = \frac{A(x+1) + B(x-5)}{(x-5)(x+1)}$

$Ax + A + Bx - 5B = 3x - 4$

$A + B = 3$ $A + B = 3$ $B = 7/6$ $A = 3 - 7/6 = 11/6$

$A - 5B = -4$ $-A + 5B = 4$

$6B = 7$

$\int \frac{11/6}{x-5} + \frac{7/6}{x+1}dx$ $\boxed{11/6 \ln|x-5| + \frac{7}{6}\ln|x+1| + C}$

e. $\int \frac{dt}{\sqrt{3-4t^2}}$

$u = 2t$

$du = 2dt$

$\frac{1}{2}du = dt$

$\frac{1}{2}\int \frac{du}{\sqrt{3-u^2}}$ $a = \sqrt{3}$

$= \frac{1}{2}\arcsin\left(\frac{u}{\sqrt{3}}\right) + C =$

$\boxed{\frac{1}{2}\arcsin\left(\frac{2t}{\sqrt{3}}\right) + C}$

15. Determine whether or not the integral converges or diverges. If the integral converges, state its value. (12 points)

$$\int_1^\infty \frac{1}{x\ln x}dx = \lim_{b\to\infty}\left[\lim_{a\to 1}\int_a^b \frac{1}{x\ln x}dx\right] =$$

$u = \ln x$

$du = \frac{1}{x}dx$

$\int \frac{1}{x\ln x}dx = \int \frac{1}{u}du$

$\ln u$

$\ln(\ln x)$

$$\lim_{b\to\infty}\left[\lim_{a\to 1} \ln(\ln b) - \ln(\ln a)\right]$$

$$\lim_{b\to\infty} \ln(\ln b) - \lim_{a\to 1} \ln(\ln a)\Big]$$

$$\lim_{b\to\infty} \ln(\ln b) - \ln(\ln 1)$$

$$\infty - (-\infty) = \infty$$

diverges

14b. $\int x\sqrt{2x+3}\,dx$ $\quad u = \sqrt{2x+3}$ $\quad u\,du = dx$

$u^2 = 2x+3$

$u^2 - 3 = 2x$

$\frac{1}{2}u^2 - \frac{3}{2} = x$

$\int\left(\frac{1}{2}u^2 - \frac{3}{2}\right)u \cdot u\,du =$

$\int \frac{1}{2}u^4 - \frac{3}{2}u^2\,du = \frac{1}{2}\cdot\frac{1}{5}u^5 - \frac{3}{2}\cdot\frac{1}{3}u^3 + C = \frac{1}{10}u^5 - \frac{1}{2}u^3 + C = \boxed{\frac{1}{10}(2x+3)^{5/2} - \frac{1}{2}(2x+3)^{3/2} + C}$

14a. $\int 2\sin\theta\cos\theta(1-2\sin^2\theta)\,d\theta = \int 2\sin\theta\cos\theta - 4\sin^3\theta\cos\theta\,d\theta$ $\quad u = \sin\theta \quad du = \cos\theta\,d\theta$

$\int 2u - 4u^3\,du = u^2 - u^4 + C = \boxed{\sin^2\theta - \sin^4\theta + C}$