6/21/2021

Implicit Differentiation and the Chain Rule (14.5)

Chain Rule

Back in Calc I: $\frac{d}{dx}\left[f\big(u(x)\big)\right] = \frac{df}{du} \cdot \frac{du}{dx} = \frac{df}{dx}$

In Multivariable calculus:

We working with functions defined with one variable (function), in terms of more than one variable; or we are working with functions defined with multiple variables (functions), each in terms of one variable; or we are working with functions defined with multiple variables (functions), each of these is defined in terms of multiple variables.

Example. $z = f(x, y)$, but $x = x(t)$ and $y = y(t)$.

$$\frac{dz}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

Example. $z = f(x, y)$, but $x = x(t, s)$ and $y = y(t, s)$.

$$\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t}$$

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s}$$

Example. $w = xe^{\frac{y}{z}} = xe^{yz^{-1}}, x = t^2, y = 1 - t, z = 1 + 2t$

$$\frac{dw}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial f}{\partial z} \cdot \frac{dz}{dt}$$

$\frac{\partial w}{\partial x} = e^{\frac{y}{z}} = e^{\frac{1-t}{1+2t}}$ $\frac{\partial w}{\partial y} = \frac{x}{z} e^{\frac{y}{z}} = \frac{t^2}{(1+2t)} e^{\frac{1-t}{1+2t}}$ $\frac{\partial w}{\partial z} = -\frac{xy}{z^2} e^{\frac{y}{z}} = -\frac{t^2}{(1+2t)^2} e^{\frac{1-t}{1+2t}}$	$\frac{dx}{dt} = 2t$ $\frac{dy}{dt} = -1$ $\frac{dz}{dt} = 2$

In the end, we want a derivative that contains only t.

$$\frac{dw}{dt} = e^{\frac{1-t}{1+2t}}(2t) + \frac{t^2}{(1+2t)} e^{\frac{1-t}{1+2t}}(-1) - \frac{t^2}{(1+2t)^2} e^{\frac{1-t}{1+2t}}(2)$$

If a value for t was available, you could plug that in here.

Example.

$$z = x^2 y^2, x = s \cos t, y = s \sin t$$

Find the partial derivatives $\frac{\partial z}{\partial t}$ and $\frac{\partial z}{\partial s}$.

$$\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t}$$

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s}$$

$\frac{\partial z}{\partial x} = 2xy^2 = 2s^3 \cos t \sin^2 t$ $\frac{\partial z}{\partial y} = 2x^2 y = 2s^3 \cos^2 t \sin t$	$\frac{\partial x}{\partial t} = -s \sin t$ $\frac{\partial x}{\partial s} = \cos t$	$\frac{\partial y}{\partial t} = s \cos t$ $\frac{\partial y}{\partial s} = \sin t$

I want my function (my final derivatives) to have only s and t.

$$\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t} = 2s^3 \cos t \sin^2 t\,(-s \sin t) + 2s^3 \cos^2 t \sin t\,(s \cos t)$$

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s} = 2s^3 \cos t \sin^2 t\,(\cos t) + 2s^3 \cos^2 t \sin t\,(\sin t)$$

Implicit Differentiation

Example with x and y only.

$$y \cos x = x^2 + y^2$$

Find $\frac{dy}{dx}$.

$$y' \cos x + y(-\sin x) = 2x + 2y\,y'$$

$$y' \cos x - 2y\,y' = 2x + y \sin x$$

$$y'(\cos x - 2y) = 2x + y \sin x$$

$$\frac{dy}{dx} = y' = \frac{(2x + y \sin x)}{\cos x - 2y}$$

Example.

$$e^z = xyz$$

Find the partial derivatives $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$.
When x,y,z are all in the equation, the function variable is z.
When x,y,z,w are all in the equation, the function variable is w.

For $\frac{\partial z}{\partial x} = z_x$, x is the derivative variable, y is constant, and z is the function (chain rule) variable.

$$e^z z_x = y(z + x z_x)$$
$$e^z z_x = yz + xyz_x$$

$$e^z z_x - xyz_x = yz$$

$$z_x(e^z - xy) = yz$$

$$\frac{\partial z}{\partial x} = z_x = \frac{yz}{e^z - xy}$$

For $\frac{\partial z}{\partial y} = z_y$, x is constant, y is the derivative variable, and z is the function (chain rule) variable.

$$e^z = xyz$$

$$e^z z_y = xz + xyz_y$$

$$e^z z_y - xyz_y = xz$$
$$z_y(e^z - xy) = xz$$

$$\frac{\partial z}{\partial y} = z_y = \frac{xz}{e^z - xy}$$

"The long way". By way of the definition.

There is a shorter way to find the implicit partial derivatives.

Create a function F that is equivalent to all the terms of the implicit function moved to one side of the equation (i.e. set equal to zero)., and then find the partial derivatives of F.

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}$$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}, \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

Example (revisited).

$$y \cos x = x^2 + y^2$$

$$F(x, y) = x^2 + y^2 - y \cos x$$

$$F_x = 2x + y \sin x$$
$$F_y = 2y - \cos x$$

$$\frac{dy}{dx} = -\frac{2x + y \sin x}{2y - \cos x}$$

From before:

$$\frac{dy}{dx} = y' = \frac{(2x + y \sin x)}{\cos x - 2y}$$

Example (revisited).

$$e^z = xyz$$

$$F(x,y,z) = xyz - e^z$$
$$F_x = yz$$
$$F_y = xz$$
$$F_z = xy - e^z$$

$$\frac{\partial z}{\partial x} = -\frac{yz}{xy - e^z}$$

$$\frac{\partial z}{\partial y} = -\frac{xz}{xy - e^z}$$

From before:

$$\frac{\partial z}{\partial x} = z_x = \frac{yz}{e^z - xy}$$

$$\frac{\partial z}{\partial y} = z_y = \frac{xz}{e^z - xy}$$

If there are fractions inside of fractions: you should get rid of them (get rid of complex fractions).

On an exam, read the problem carefully. If I don't specify a method, you can use the short-cuts. If I ask you to verify, then you'll have to do it both ways and show that they agree.

Relative Extrema (14.7)

Critical Points (where possible extrema can occur) can be of 4 possible flavors:
Local Maxima, Local Minima, Undetermined, Saddle Points (new one)

The critical points occur when the gradient (∇f) is the zero vector: the same place where all of the partial derivatives are zero at the same time.

We can use level curve graphs or gradient fields to identify critical points. Next time, we will find them algebraically from the partial derivatives, and use the second partials test to determine the type of critical point it is (if it can be determined).

There are handouts on chain rule, on implicit differentiation, and extrema.

I will postpone HW 10 for another day.