6/2/2021

Triple Integrals (15.7, 15.8, 15.9)

Like double integrals, but we are using three variables instead of 2.
Instead of dA, we use dV

$$dA = dydx = dxdy = rdrd\theta$$
$$dV = dzdydx = dzdxdy = dydzdx = dydxdz = dxdzdy = dxdydz$$

Inside integral can have variables in the limits as long we are not integrating at that step, and haven't integrated before. The outer one can only be constants.

$$\int_a^b \int_{f(x)}^{g(x)} \int_{F(x,y)}^{G(x,y)} dz\,dy\,dx$$

$$\int_a^b \int_{f(y)}^{g(y)} \int_{F(y,z)}^{G(y,z)} dx\,dz\,dy$$

And so forth.

$\iiint_Q dV$ = volume
$\iiint_Q f(x,y,z)dV$ = 4D volume, if f is a density function, we could think as the total mass; if f is a probability density function, then integrating produces the total probability of the region.

Example.

$$\int_0^{\sqrt{\pi}} \int_0^x \int_0^{xz} x^2 \sin y\,dy\,dz\,dx$$

$$region: y = 0, y = xz, z = 0, z = x, x = 0, x = \sqrt{\pi}$$

$$\int_0^{\sqrt{\pi}} \int_0^x x^2 \left(-\cos y\right)\Big|_{y=0}^{y=xz} dzdx = \int_0^{\sqrt{\pi}} \int_0^x -x^2 \cos xz + x^2 dzdx$$

$$\int_0^{\sqrt{\pi}} -x \sin xz + x^2 z|_{z=0}^{z=x}\,dx = \int_0^{\sqrt{\pi}} -x \sin x^2 + x^3 dx$$

$$\frac{1}{2}\cos x^2 + \frac{x^4}{4}\Big|_0^{\sqrt{\pi}} = -\frac{1}{2} + \frac{\pi^2}{4} - \frac{1}{2} = \frac{\pi^2}{4} - 1$$

Example.

$$\iiint\limits_Q \frac{z}{x^2 + z^2} dV$$

$$Q = \{(x, y, z) | 1 \le y \le 4, y \le z \le 4, 0 \le x \le z\}$$

$$\int_1^4 \int_y^4 \int_0^z \frac{z}{x^2 + z^2} \, dx \, dz \, dy$$

Start with inverse tangent.

Example.

$$\iiint_Q 6xydV$$

Region: lies under the plane $z = 1 + x + y$ and above the region in the xy-plane (z=0) bounded by the curves $y = \sqrt{x}, y = 0, x = 1$.

$$\int_0^1 \int_0^{\sqrt{x}} \int_0^{1+x+y} 6xydzdydx$$

Example.

$$\iiint_Q zdV$$

Region: bounded by the cylinder $y^2 + z^2 = 9$ and the planes $x = 0, y = 3x \to x = \frac{y}{3}, z = 0$

(one option is to do a change of variables: swapping variables for each other in order to re-orient the surfaces so that they are easier to visualize or graph in a program—if you do though, you must do the same transformation on all the variables in the entire problem, including the function to be integrated.)

Suppose I swapped x with z.
Region: $y^2 + x^2 = 9, z = 0, y = 3z, x = 0, \iiint_Q xdV$

$$\int_0^3 \int_0^{\sqrt{9-y^2}} \int_0^{y/3} z \, dxdzdy$$

Example.

$$\iiint_Q xdV$$

Region: bounded by the paraboloid $x = 4y^2 + 4z^2, x = 4$

Equivalent to:

$$\iiint_Q zdV$$

Region: bounded by the paraboloid $z = 4y^2 + 4x^2, z = 4$

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{4y^2+4x^2}^{4} zdzdy\,dx$$

Intersect: $4 = 4y^2 + 4x^2 \rightarrow x^2 + y^2 = 1$

Recall that cylindrical is basically just polar + z
Limits for the z integral can be swapped into polar algebraically, as can any functions to be integrated.
Only the x and y integrals need to switched graphically.

$$4y^2 + 4x^2 = 4(x^2 + y^2) = 4r^2$$

$$\int_{0}^{2\pi}\int_{0}^{1}\int_{4r^2}^{4} zrdzdrd\theta$$

dV in cylindrical is essentially $dzdA = dz(rdrd\theta) = rdzdrd\theta$

Paraboloids, cones, anything in polar (planar regions defined by polar graphs or circles), sometimes spheres (depends on what they are intersecting) usually are decent in cylindrical.

Hyperboloids, hyperbolic paraboloids may be better in cylindrical than in rectangular.

We'll see later, that the combination of cones and spheres are usually great in spherical.

Example.

$$\iiint_Q dV$$

Region: between the planes $z = 0, z = x + y + 5$, bonded by the cylinders $x^2 + y^2 = 4, x^2 + y^2 = 9$.

$$r = 2, r = 3, 0 \le \theta \le 2\pi, z = 0, z = r\cos\theta + r\sin\theta + 5$$

$$\int_{0}^{2\pi}\int_{2}^{3}\int_{0}^{r\cos\theta + r\sin\theta+5} rdzdr\,d\theta$$

Example.
Find the volume bounded by $z = x^2 + y^2, z = 36 - 3x^2 - 3y^2$

$$x^2 + y^2 = 36 - 3x^2 - 3y^2$$
$$4x^2 + 4y^2 = 36$$
$$x^2 + y^2 = 9$$

$$\int_0^{2\pi}\int_0^3\int_{r^2}^{36-3r^2} r\,dz\,dr\,d\theta$$

Example.

$$\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_{\sqrt{x^2+y^2}}^{2} xz\,dz\,dx\,dy$$

$$\int_0^{2\pi}\int_0^2\int_r^2 (r\cos\theta)z\,r\,dz\,dr\,d\theta$$

Spherical $dV = \rho^2 \sin\phi \, d\rho d\phi d\theta$

When we transform from rectangular to spherical, all the limits of the integrals have to be transformed graphically. Only the function to be integrated can be swapped out algebraically.

For next class to review for the exam:
Pick some problems you'd like me to do for you/with you.