6/14/2021

Surface Integrals (16.7)

Two kinds of surface integrals.
When we did line integrals: we functions integrated over a curve, and we had field integrated over a curve (field, or differential form). We have something similar for surfaces. There are surface integrals over a function, and surface integrals integrated over a field (flux).

Surface integral over a function: (think of $z = g(x, y)$ as the surface

$$\iint_R f(x,y,z)dS = \iint_R f(x,y,z)\|\nabla G\|\, dA = \iint_R f\big(x,y,g(x,y)\big)\sqrt{g_x^2 + g_y^2 + 1}dA$$

$$\iint_R f(x,y,z)dS = \iint_R f(x,y,z)\|\mathrm{N}\|\, dA = \iint_R f\big(x(u,v),y(u,v),z(u,v)\big)\|r_u \times r_v\|dA$$

Example.

$$\iint_R (x+y+z)dS$$

Over the surface S is the parallelogram with parametric equations $x = u + v, y = u - v, z = 1 + 2u + v, 0 \le u \le 2, 0 \le v \le 1$.

$$r(u,v) = \langle u+v, u-v, 1+2u+v \rangle$$

$$r_u = \langle 1,1,2 \rangle$$
$$r_v = \langle 1,-1,1 \rangle$$

$$r_u \times r_v = \begin{vmatrix} i & j & k \\ 1 & 1 & 2 \\ 1 & -1 & 1 \end{vmatrix} = (1+2)i - (1-2)j + (-1-1)k = \langle 3,1,-2 \rangle$$

$$\|r_u \times r_v\| = \sqrt{9+1+4} = \sqrt{14}$$

$$\int_0^1 \int_0^2 \big((u+v) + (u-v) + (1+2u+v)\big)\sqrt{14}dudv = \int_0^1 \int_0^2 (4u+1+v)\sqrt{14}dudv$$

$$\int_0^1 2u^2 + u + uv\Big|_0^2 dv = \int_0^1 10 + 2vdv = 10v + v^2|_0^1 = 11$$

If we think of the function we are integrating as the (mass density) of the surface, then we can think of the result as the total mass of the surface.

Example.

$$\iint_R xz dS$$

S is the part of the plane $2x + 2y + z = 4$ that lies in the first octant.

$$z = 4 - 2x - 2y$$

$$\iint_R x(4 - 2x - 2y) dS$$

$$G(x, y, z) = 2x + 2y + z - 4$$
$$\nabla G = \langle 2,2,1 \rangle$$
$$\|\nabla G\| = \sqrt{4 + 4 + 1} = 3$$

$$\int_0^2 \int_0^{2-x} (4x - 2x^2 - 2xy) 3 dy dx$$
$$0 = 4 - 2x - 2y$$
$$2y = 4 - 2x$$
$$y = 2 - x$$
$$0 = 2 - x$$
$$x = 2$$
$$\int_0^2 \int_0^{2-x} (4x - 2x^2 - 2xy) 3 dy dx = \int_0^2 4xy - 2x^2 y - xy^2 \Big|_0^{2-x} dx =$$

$$\int_0^2 4x(2 - x) - 2x^2(2 - x) - x(2 - x)^2 dx = \int_0^2 8x - 4x^2 - 4x^2 + 2x^3 - 4x + 4x^2 - x^3 dx =$$
$$(2 - x)^2 = 4 - 4x + x^2$$

$$\int_0^2 4x - 4x^2 + x^3 dx = 2x^2 - \frac{4}{3}x^3 + \frac{1}{4}x^4 \Big|_0^2 = 8 - \frac{32}{3} + 4 = \frac{4}{3}$$

Example.

$$\iint_R y^2 dS$$

S is the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies inside the cylinder $x^2 + y^2 = 1$ and above the xy-plane.

$$z = \sqrt{4 - x^2 - y^2}$$

$$G(x, y, z) = \sqrt{4 - x^2 - y^2} - z$$
$$\nabla G = \langle \frac{-x}{\sqrt{4 - x^2 - y^2}}, \frac{-y}{\sqrt{4 - x^2 - y^2}}, -1 \rangle$$

$$\|\nabla G\| = \sqrt{\frac{x^2}{4-x^2-y^2}+\frac{y^2}{4-x^2-y^2}+1} = \sqrt{\frac{x^2+y^2+4-x^2-y^2}{4-x^2-y^2}} = \sqrt{\frac{4}{4-x^2-y^2}}$$
$$= 2(4-x^2-y^2)^{-\frac{1}{2}}$$

$$\iint_R y^2 dS = \iint_R y^2 \cdot 2(4-x^2-y^2)^{-\frac{1}{2}} dA = 2\int_0^{2\pi}\int_0^1 \frac{r^2\sin^2\theta}{\sqrt{4-r^2}} r dr d\theta$$

$$= 2\int_0^{2\pi} \sin^2\theta \, d\theta \int_0^1 \frac{r^3}{\sqrt{4-r^2}} dr$$

$$2\int_0^{2\pi} \frac{1}{2}(1-\cos 2\theta) d\theta = \int_0^{2\pi} 1-\cos 2\theta \, d\theta = \theta - \frac{1}{2}\sin 2\theta\Big|_0^{2\pi} = 2\pi$$

$$\int_0^1 \frac{r^3}{\sqrt{4-r^2}} dr$$

$$u = r^2, dv = \frac{r}{\sqrt{4-r^2}} dr, du = 2rdr, dv = -\frac{1}{2}p^{-\frac{1}{2}}, v = -p^{\frac{1}{2}} = -\sqrt{4-r^2}$$

$$p = 4-r^2, dp = -2rdr \rightarrow -\frac{1}{2}dp = rdr$$

$$\int_0^1 \frac{r^3}{\sqrt{4-r^2}} dr = -r^2\sqrt{4-r^2}\Big|_0^1 + \int_0^1 2r\sqrt{4-r^2} dr = -r^2\sqrt{4-r^2} - \frac{2}{3}(4-r^2)^{\frac{3}{2}}\Big|_0^1$$
$$= -1(\sqrt{3}) - \frac{2}{3}(3\sqrt{3}) + \frac{2}{3}(8) = \frac{16}{3} - 3\sqrt{3}$$

$$2\int_0^{2\pi} \sin^2\theta \, d\theta \int_0^1 \frac{r^3}{\sqrt{4-r^2}} dr = 2\pi\left(\frac{16}{3} - 3\sqrt{3}\right)$$

In parametric form $r(u,v) = \langle 2\cos u \sin v, 2\sin u \sin v, 2\cos v\rangle$

$$0 \le u \le 2\pi, 0 \le v \le \frac{\pi}{6}$$

$$z^2 + y^2 + z^2 = 4 \rightarrow \rho = 2$$
$$x^2 + y^2 = 1 \rightarrow \rho^2 \sin^2\phi = 1 \rightarrow \rho\sin\phi = 1 \rightarrow \rho = \csc\phi$$
$$\csc\phi = 2 \rightarrow \sin\phi = \frac{1}{2} \rightarrow \phi = \frac{\pi}{6}$$

I won’t finish this here, but this is another option for solving the problem.

The second kind of surface integral uses a field rather than a single expression for a function.

$$\iint_R \vec{F}(x,y,z)\cdot d\vec{S} = \iint_R F(x,y,z)\cdot \vec{N}dA = \iint_R F(x,y,z)\cdot \nabla G dA = \iint_R F(x,y,z)\cdot (r_u \times r_v)dA$$

Oriented surfaces:

We have to think about the sign of the normal vector.

$$z = 4 - 2x - 2y$$
$$G(x,y,z) = 2x + 2y + z - 4 \ \rightarrow \nabla G = \langle 2,2,1\rangle$$
$$H(x,y,z) = 4 - 2x - 2y - z \rightarrow \nabla H = \langle -2,-2,-1\rangle$$

Example.

$$\iint_R F\cdot d\vec{S}$$

Over the surface S is the parallelogram with parametric equations $x = u + v, y = u - v, z = 1 + 2u + v, 0 \le u \le 2, 0 \le v \le 1$.

$$r(u,v) = \langle u+v, u-v, 1+2u+v\rangle$$

$$r_u = \langle 1,1,2\rangle$$
$$r_v = \langle 1,-1,1\rangle$$

$$r_u \times r_v = \begin{vmatrix} i & j & k \\ 1 & 1 & 2 \\ 1 & -1 & 1 \end{vmatrix} = (1+2)i - (1-2)j + (-1-1)k = \langle 3,1,-2\rangle$$

$$d\vec{S} = \langle 3,1,-2\rangle dA$$

This normal vector has a downward orientation. If we need the upward orientation, flip all the signs.

$$F(x,y,z) = \langle ze^{xy}, -3ze^{xy}, xy\rangle$$
$$\iint_R \langle ze^{xy}, -3ze^{xy}, xy\rangle\cdot\langle 3,1,-2\rangle dA = \iint_R 3ze^{xy} - 3ze^{xy} - 2xydA =$$

$$\int_0^1\int_0^2 -2(u+v)(u-v)dudv = \int_0^1\int_0^2 -2u^2 + 2v^2dudv = \int_0^1 -\frac{2}{3}u^3 - 2v^2u\Big|_0^2 dv =$$

$$\int_0^1 -\frac{16}{3} - 4v^2dv = -\frac{16}{3}v - \frac{4}{3}v^3\Big|_0^1 = -\frac{16}{3} - \frac{4}{3} = -\frac{20}{3}$$

What does the sign mean?

In the direction of the normal vector, the flow is net negative (flow is in the opposite direction of the normal vector).

Sinks, Sources or Incompressible Flows for closed volume

Sinks: have a net negative flow through the volume
Sources: have a net positive flow through the volume
Incompressible: net flow is zero

Closed surface boundaries: default orientation is "outward" from the interior of the surface.

Example.

$$\iint_R F \cdot d\vec{S}$$

Surface is part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the square $0 \le x \le 1, 0 \le y \le 1$ and has upward orientation.

$$G(x, y, z) = z - 4 + x^2 + y^2$$
$$\nabla G = \langle 2x, 2y, 1 \rangle$$

$$F(x, y, z) = \langle xy, yz, zx \rangle$$

$$\iint_R F \cdot d\vec{S} = \int_0^1 \int_0^1 \langle xy, y(4 - x^2 - y^2), (4 - x^2 - y^2)x \rangle \cdot \langle 2x, 2y, 1 \rangle dydx$$

$$\int_0^1 \int_0^1 2x^2y + 8y^2 - 2x^2y^2 - 2y^4 + 4x - x^3 - xy^2 dydx =$$

$$\int_0^1 x^2y^2 + \frac{8}{3}y^3 - \frac{2}{3}x^2y^3 - \frac{2}{5}y^5 + 4xy - x^3y - \frac{1}{3}xy^3 \Big|_0^1 dx =$$

$$\int_0^1 x^2 + \frac{8}{3} - \frac{2}{3}x^2 - \frac{2}{5} + 4x - x^3 - \frac{1}{3}xdx = \int_0^1 \frac{4}{3}x^2 + \frac{34}{15} + \frac{11}{3}x - x^3 dx =$$

$$\frac{4}{9}x^3 + \frac{34}{15}x + \frac{11}{6}x^2 - \frac{1}{4}x^4 = \frac{4}{9} + \frac{34}{15} + \frac{11}{6} - \frac{1}{4} = \frac{773}{180}$$

Next time:
Flow through closed surfaces, maybe set up the integrals for one example with multiple surface boundaries. Apply the Divergence Theorem to simplify.