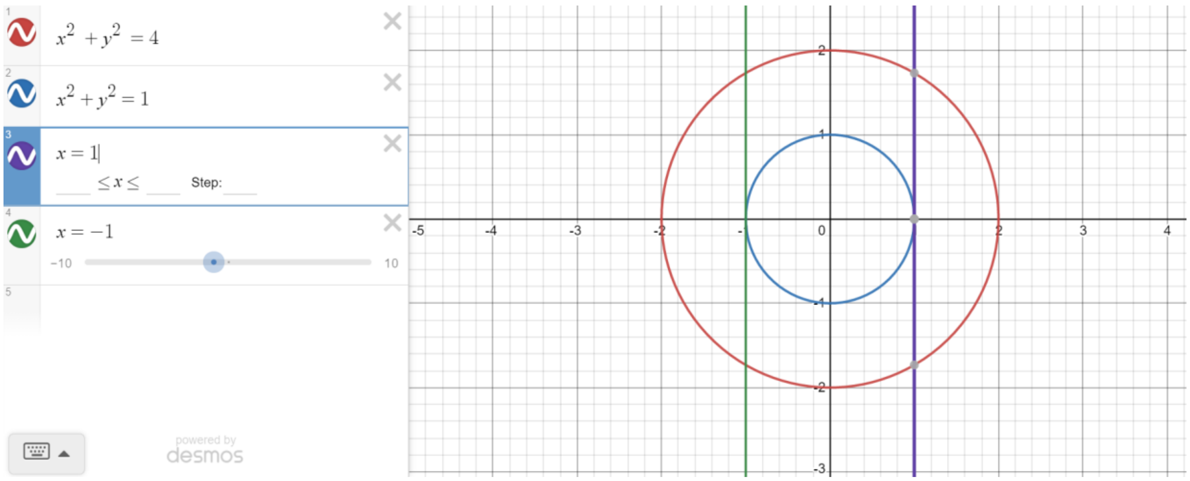
6/1/2021

Double integrals in Polar Coordinates

Polar coordinates covers the xy-plane using radius (distance from the origin = r) and an angle (θ relative to the positive x-axis) to express coordinates.

In polar, coordinates of points are not unique. We are going to use properties of symmetry.

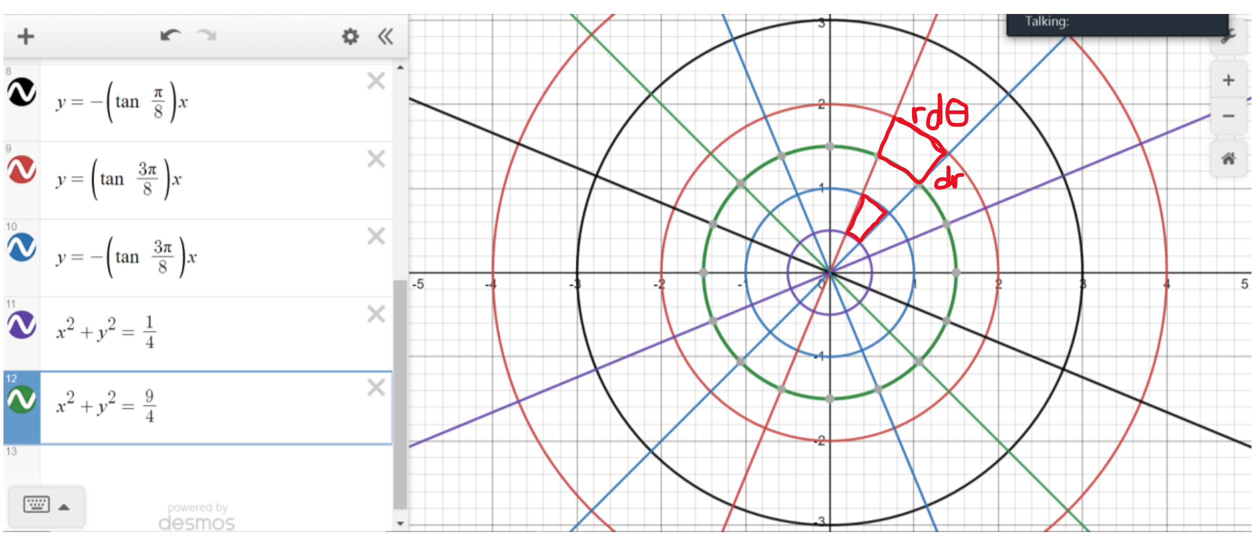
A circle in polar is $r = constant$
An annulus is a circle with a hole in it



This would require 4 separate integrals (and even if we use symmetry, we'd need 2 of them) and use trig substitution. Polar allows us to avoid all of that.

These two circles are just r=1, r=2. And the inner and outer functions in polar are always the same, so they can be done in one integral.

One complication to switching to polar in two dimensions.



$dA = dydx \; or \; dxdy$ in rectangular

$$dA = rdrd\theta$$

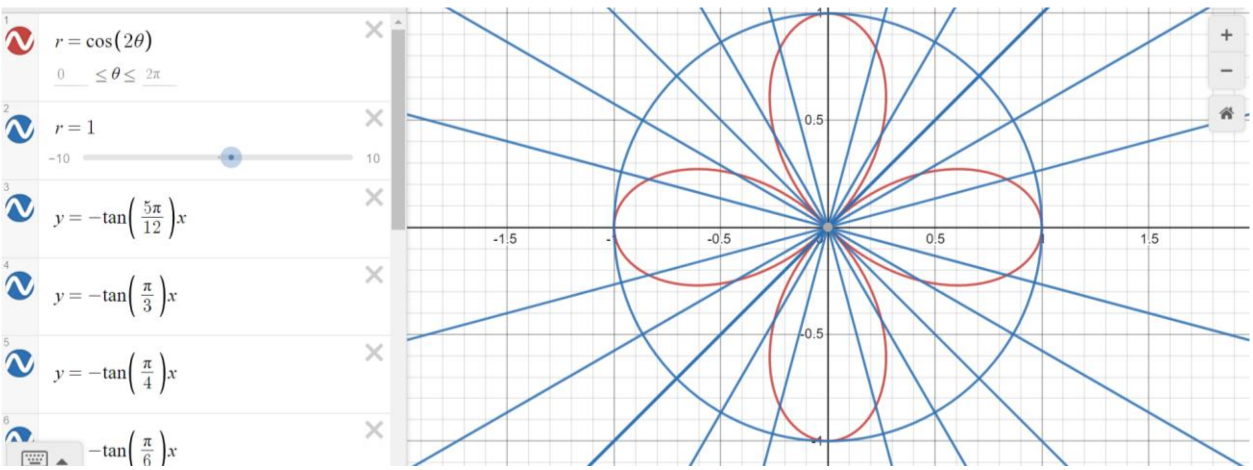
$$\iint_R dA = \iint_R rdrd\theta = \int_{R(\theta)} \frac{1}{2} r^2 d\theta$$

$$\int_{\theta_1}^{\theta_2} \int_{r_{inner}}^{r_{outer}} rdrd\theta = \int_{\theta_1}^{\theta_2} \frac{1}{2} r_{outer}^2 - \frac{1}{2} r_{inner}^2 d\theta$$

Example already in polar.

Find the area using a double integral of one petal of the rose $r = \cos 2\theta$.

https://www.desmos.com/calculator/ms3eghkkgz



Rightmost lobe, using symmetry from 0 to $\frac{\pi}{4}$, times 2

$$\cos 2\theta = 0$$
$$2\theta = \frac{\pi}{2}, -\frac{\pi}{2}, \frac{3\pi}{2}, -\frac{3\pi}{2}, etc.$$
$$\theta = \frac{\pi}{4}, -\frac{\pi}{4}, etc.$$

$$2\int_0^{\frac{\pi}{4}} \int_0^{\cos 2\theta} rdrd\theta = 2\int_0^{\frac{\pi}{4}} \frac{1}{2} r^2 \Big|_0^{\cos 2\theta} d\theta = 2 \cdot \frac{1}{2} \int_0^{\frac{\pi}{4}} \cos^2 2\theta \, d\theta =$$

$$2 \cdot \frac{1}{2} \cdot \frac{1}{2} \int_0^{\frac{\pi}{4}} 1 + \cos 4\theta \, d\theta = \frac{1}{2}\left[\theta + \frac{1}{4}\sin 4\theta\right]_0^{\frac{\pi}{4}} = \frac{1}{2}\left[\frac{\pi}{4} + 0\right] = \frac{\pi}{8}$$

Examples switching from rectangular to polar.

$$\iint_R (2x - y)dA$$

The region is in the first quadrant enclosed in the circle $x^2 + y^2 = 4$, and the lines x=0, and y=x.

In rectangular (area only):

$$\int_0^{\sqrt{2}} \int_x^{\sqrt{4-x^2}} dydx = \int_0^{\sqrt{2}} \sqrt{4 - x^2} - xdx$$

In polar (area only):

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^2 rdrd\theta$$

For the volume, use the function in the appropriate coordinate system. Switch with algebraic substitution.

Volume in rectangular:

$$\int_0^{\sqrt{2}} \int_x^{\sqrt{4-x^2}} (2x - y)dydx$$

Volume in polar:

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^2 (2rcos\,\theta - r\sin\theta)rdrd\theta$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^2 (2r^2cos\,\theta - r^2\sin\theta)drd\theta$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{2}{3}r^3\cos\theta - \frac{1}{3}r^3\sin\theta\Big|_0^2 \, d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{16}{3}\cos\theta - \frac{8}{3}\sin\theta\, d\theta$$

$$\frac{16}{3}\sin\theta + \frac{8}{3}cos\theta\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \frac{16}{3}\left[1 - \frac{1}{\sqrt{2}}\right] + \frac{8}{3}\left[0 - \frac{1}{\sqrt{2}}\right] = \frac{16}{3} - \frac{16}{3\sqrt{2}} - \frac{8}{3\sqrt{2}} = \frac{16}{3} - \frac{24}{3\sqrt{2}}$$

$$\iint_R \frac{x^2}{x^2 + y^2} dA$$

Example.

Below the paraboloid $z = 18 - 2x^2 - 2y^2$ and above the xy-plane.

$$0 = 18 - 2x^2 - 2y^2$$
$$x^2 + y^2 = 9$$

$$z = 18 - 2x^2 - 2y^2 = 18 - 2(x^2 + y^2) = 18 - 2r^2$$

$$\int_0^{2\pi} \int_0^3 (18 - 2r^2) r dr d\theta$$

When you are given a scenario that needs to be integrated in polar coordinates, must graphically convert the limits of integration, but algebraically convert the function.

Example.

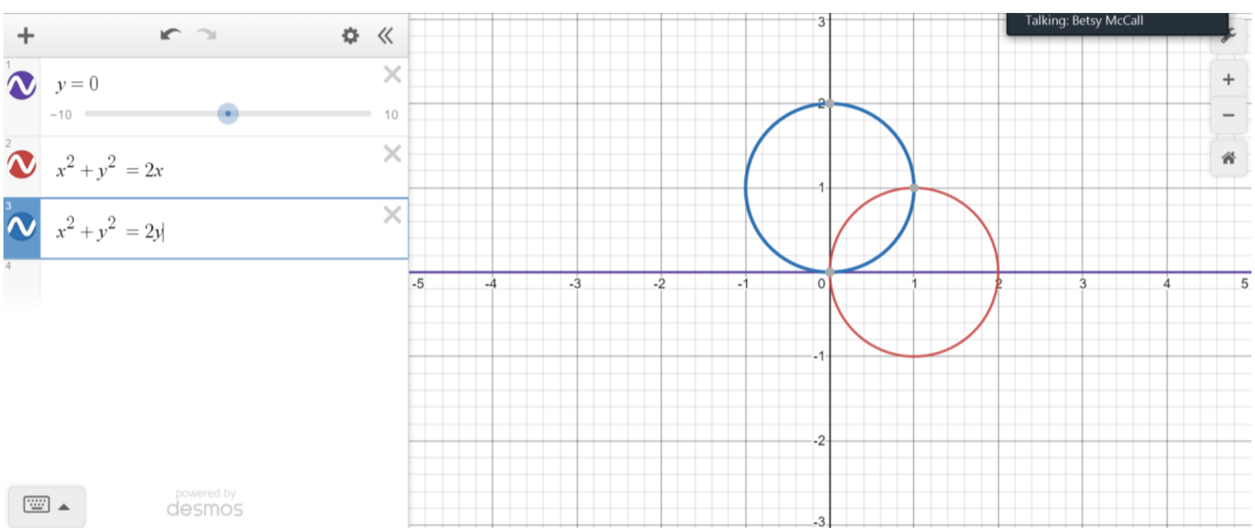
$$\int_0^1 \int_y^{\sqrt{2-y^2}} (x + y) dx dy$$

$$y = 1, y = 0, x = y, x = \sqrt{2 - y^2}$$
$$x^2 = 2 - y^2 \rightarrow x^2 + y^2 = 2$$

$$\int_0^{\frac{\pi}{4}} \int_0^{\sqrt{2}} (r\cos\theta + r\sin\theta) r dr d\theta$$

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2 + y^2} dy dx$$

$$y = \sqrt{2x - x^2}$$
$$y^2 = 2x - x^2$$
$$x^2 + y^2 = 2x$$
$$r^2 = 2r\cos\theta$$
$$r = 2\cos\theta$$
$$r = 2a\cos\theta$$



The limits of integration in θ for these kinds of circles are from 0 to π instead of 0 to 2π.

$$\int_0^{\pi}\int_0^{2\cos\theta}(r)r\,dr\,d\theta$$

$$\cos^3\theta = \cos\theta\,(1-\sin^2\theta)$$