5/25/2021

Review for Exam
Gradients/level curves/traces
Conservative vector fields/potential functions?

Want to be able to visualize three-dimensional functions even though we can only draw them in 2D.

One way to look at a 3D function is the trace: sets x or y equal to a constant (generally zero), and then see how that function/surface intersects with the plane.

If we set x=0, then the graph, function that remains is the intersection of the surface with the yz-plane
If we set y=0, then the graph/function that remains is the intersection of the surface with xz-plane.
Let z=vertical axis.

$$f(x,y) = (y-2x)^2$$

Trace (x=0) $z = y^2$
Trace (y=0) $z = 4x^2$

$$f(x,y) = \frac{y}{x^2+y^2}$$

Trace (x=0) $z = \frac{1}{y}$
Trace (y=0) $z = 0$

Level Curves/Contour Curves

Looking down on the xy-plane from the positive z direction
Is to let z=constant (set a constants), and then draw the resulting curve in the xy-plane, with z set equal to these different constants.

$$f(x,y) = (y-2x)^2$$

$$c = (y-2x)^2$$
$$\pm\sqrt{c} = y - 2x$$
$$y = \pm\sqrt{c} + 2x$$

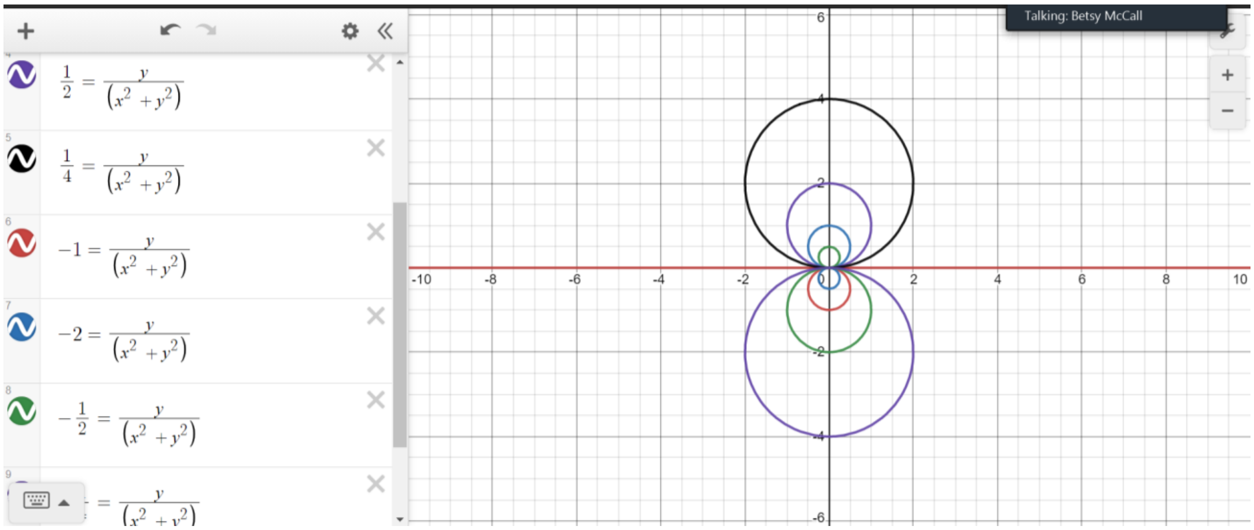
$$f(x,y) = \frac{y}{x^2+y^2}$$

$$c = \frac{y}{x^2+y^2}$$
$$x^2 + y^2 = \frac{y}{c}$$
$$y^2 - \frac{y}{c} = -x^2$$

$$y^2 - \frac{y}{c} + \frac{1}{4c^2} = -x^2 + \frac{1}{4c^2}$$

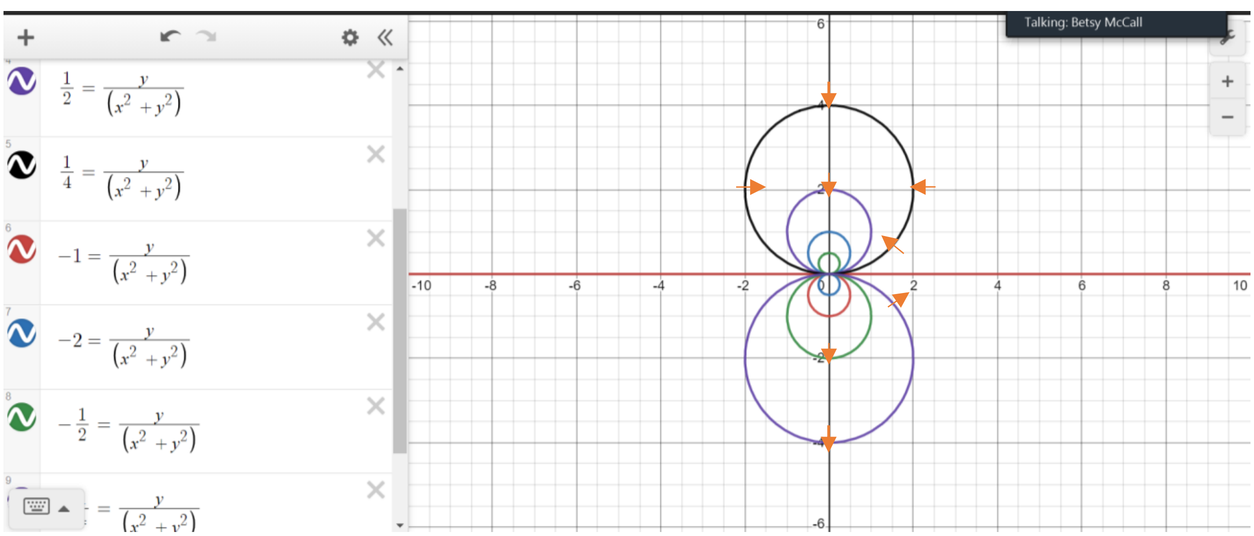
$$\left(y - \frac{1}{c}\right)^2 = \frac{1}{4c^2} - x^2$$

$$y = \frac{1}{c} \pm \sqrt{\frac{1}{4c^2} - x^2}$$



How is this related to gradients?

The level curves are perpendicular to the vector field of the gradient.
The relationship goes both ways. You can use level curves to plot the gradient field.
You can use the gradient field to sketch the level curves.



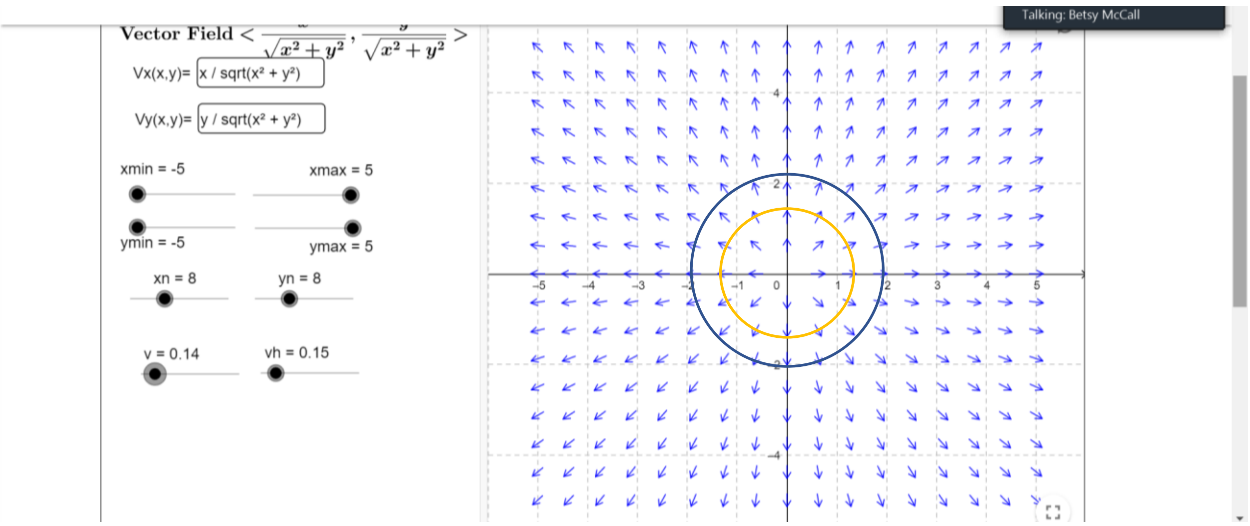
$$f(x, y) = \sqrt{x^2 + y^2}$$

Top half of a circular cone

What is the gradient of this function? ∇f?

$$\langle \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}}(2x), \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}}(2y)\rangle$$

$$\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\rangle$$



Conservative vector fields and potential functions

A vector field is conservative if the curl of the vector field is identically zero for all values of x, y, and z.

$$\nabla \times F = \vec{0}$$

If a field is conservative, then there exists a potential function f such that $F = \nabla f$.

$$f(x,y,z) = x^2 + xy + 3y^2 + yz - 4z^2$$

$$\nabla f = \langle 2x + y, x + 6y + z, y - 8z\rangle = \langle f_x, f_y, f_z\rangle$$

$$\nabla \times \nabla f = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x + y & x + 6y + z & y - 8z \end{vmatrix} =$$

$$(1-1)i - (0-0)j + (1-1)k = \langle 0,0,0\rangle$$
$$(f_{zy} - f_{yz})i - (f_{zx} - f_{xz})j + (f_{yx} - f_{xy})k = 0$$

$$F(x,y,z) = \langle y^2z^3 + 1, 2xyz^3 - 2 + z, 3xy^2z^2 + \sin z + y\rangle = \langle M, N, P\rangle$$

Is this field conservative? If so, find the potential function.

$$\nabla \times \mathrm{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2z^3 + 1 & 2xyz^3 - 2 + z & 3xy^2z^2 + \sin z + y \end{vmatrix} =$$

$$\left(6xyz^2 + 1 - (6xyz^2 + 1)\right)i - (3y^2z^2 - 3y^2z^2)j + (2yz^3 - 2yz^3)k = 0$$

This means the field is conservative and there is a potential function. What is it?

Want to find $\phi(x, y, z)$.

$M = \frac{\partial \phi}{\partial x} \to \int M\, dx$ gives me terms of ϕ that contain x
$N = \frac{\partial \phi}{\partial y} \to \int N\, dy$ gives me terms of ϕ that contain y
$P = \frac{\partial \phi}{\partial z} \to \int P dz$ gives me terms of ϕ that contain z

Terms that contain all the variables will appears in all the antiderivatives
Terms that contain only two of the variables will appear in two of the antiderivatives
Terms that contain only one of the variables will appear in only the one antiderivative
Constants can't be reconstructed

$$\int y^2z^3 + 1\, dx = xy^2z^3 + x + f(y, z)$$
$$\int 2xyz^3 - 2 + z\, dy = xy^2z^3 - 2y + yz + g(x, z)$$
$$\int 3xy^2z^2 + \sin z + y\, dz = xy^2z^3 - \cos z + yz + h(x, y)$$

$$\phi(x, y, z) = xy^2z^3 + yz + x - 2y - \cos z + K$$

This is the potential function.

Websites used today:
https://c3d.libretexts.org/CalcPlot3D/index.html
https://www.geogebra.org/m/QPE4PaDZ
https://www.desmos.com/calculator