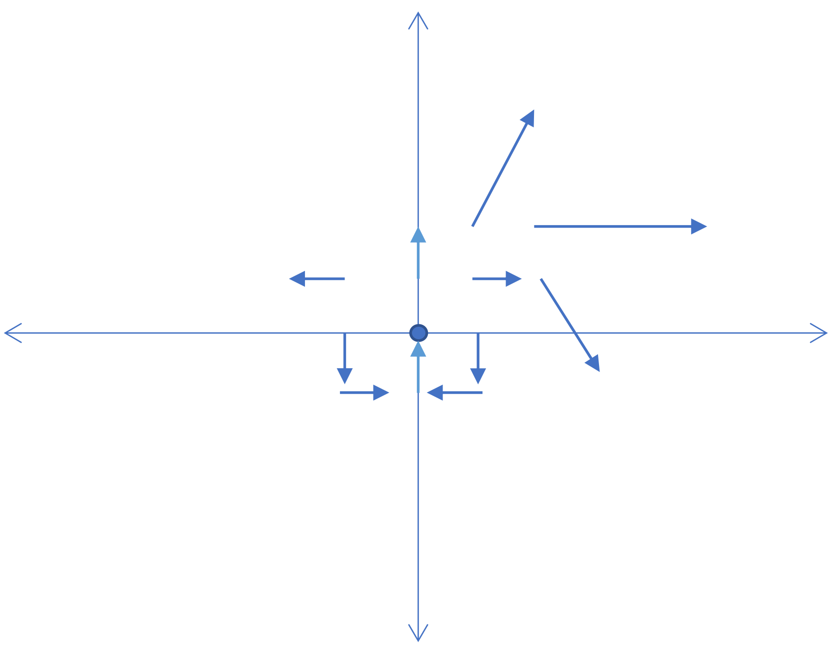
5/20/2021

Vector Fields (16.1)

Is a vector function, defined in more than one variable, that describes the magnitude and direction of a vector associated with every point in space (2D, or 3D or higher dimensions).

$$\vec{F}(x,y) = \langle xy, y^2 - x^2 \rangle$$

$$F(0,0) = \langle 0,0 \rangle$$
$$F(0,1) = \langle 0,1 \rangle$$
$$F(1,0) = \langle 0,-1 \rangle$$
$$F(0,-1) = \langle 0,1 \rangle$$
$$F(-1,0) = \langle 0,-1 \rangle$$
$$F(1,1) = \langle 1,0 \rangle$$
$$F(1,-1) = \langle -1,0 \rangle$$
$$F(-1,1) = \langle -1,0 \rangle$$
$$F(-1,-1) = \langle 1,0 \rangle$$
$$F(2,1) = \langle 2,-3 \rangle$$
$$F(1,2) = \langle 2,3 \rangle$$
$$F(2,2) = \langle 4,0 \rangle$$



https://www.desmos.com/calculator/eijhparfmd
https://www.geogebra.org/m/u3xregNW

16.2 Line Integrals

Line integral: one application of a line integral is to find the work done moving along a path (defined by a vector-valued function) through a force field (vector field).

$\int_C f(x,y)ds$ (version 1)   a function describes the force in space (mass density, etc.), integrating along a path

$$x = x(t), y = y(t)$$
$$C = r(t) = \langle x(t), y(t) \rangle$$
$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Version 2 – vector field F defines the force

$$\int_C \vec{F} \cdot d\vec{r} = \int_C F \cdot r'(t)dt$$

Make substitutions for x and y (and z) in the equation for expressions in t from the path.

$$F = \langle xy, y^2 - x^2 \rangle$$
$$C = r(t) = \langle t, 2t - 1 \rangle$$
$$F = \langle 2t^2 - t, 3t^2 - 4t + 1 \rangle$$

Limits of integration depend on the limits of t in parametrized curve

Version 3 is actually related directly to version 2, but with different notation (differential form)

$$\int_C Pdx + Qdy + Rdz$$

Vector field $F = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle$

$$dr = r'(t)dt$$
$$r(t) = \langle x(t), y(t), z(t) \rangle$$
$$r'(t) = \langle x'(t), y'(t), z'(t) \rangle$$
$$dx = x'(t)dt, dy = y'(t)dt, dz = z'(t)dt$$

Examples.

$\int_C xy^4 ds$, C is the right half of the circle $x^2 + y^2 = 16$

$$r(t) = \langle 4\cos t, 4\sin t \rangle$$
$$t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (4\cos t)(4\sin t)^4 \sqrt{(-4\sin t)^2 + (4\cos t)^2} dt$$
$$4^5 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos t \sin^4 t \sqrt{16\sin^2 t + 16\cos^2 t}\, dt$$
$$4^5 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos t \sin^4 t \sqrt{16} dt$$
$$4^6 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos t \sin^4 t\, dt$$

$$u = \sin t\,, du = \cos t\, dt$$

$$4^6 \int_{-1}^{1} u^4 du = \frac{4^6 u^5}{5}\bigg|_{-1}^{1} = \frac{4^6}{5}(2) = 1638.4$$

Evaluate the line integral $\int_C F \cdot dr$ for the given F and r(t).

$$F(x, y) = \langle xy, 3y^2 \rangle, r(t) = \langle 11t^4, t^3 \rangle, 0 \leq t \leq 1$$

$$r'(t) = \langle 44t^3, 3t^2 \rangle$$
$$F(t) = \langle 11t^7, 3t^6 \rangle$$
$$F \cdot dr = F \cdot r'(t)dt = 484t^{10} + 9t^8$$

$$\int_0^1 484t^{10} + 9t^8 dt = 44t^{11} + t^9|_0^1 = 44 + 1 = 45$$

$$\int_C (y + z)dx + (x + z)dy + (x + y)dz$$

C is the line segments from (0,0,0) to (1,0,1), then to (0,1,2)

$$r_1(t) = \langle t, 0, t \rangle, 0 \leq t \leq 1$$

$$\int_0^1 (0 + t)(1)dt + (t + t)(0)dt + (t + 0)(1)dt = \int_0^1 2tdt = t^2|_0^1 = 1$$

$$r_2(t) = \langle 1 - t, t, 1 + t \rangle, 0 \leq t \leq 1$$
$$v = \langle -1,1,1 \rangle$$

$$\int_0^1 (t + 1 + t)(-1)dt + (1 - t + 1 + t)(1)dt + (1 - t + t)(1)dt$$
$$\int_0^1 [(2t + 1)(-1) + (2) + 1]dt$$
$$\int_0^1 (-2t + 2)dt = -t^2 + 2t|_0^1 = -1 + 2 = 1$$

The final line integral value is 1+1 = 2.

Partial Derivatives (14.3)

One variable derivatives: our definition was based on difference quotient, took the limit as the points got closer together

$$f'(x) = \frac{df}{dx} = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

In multiple variables, we can only do in one variable at a time

$$f_x(x,y) = \frac{\partial f}{\partial x} = \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}$$

$$f_y(x,y) = \frac{\partial f}{\partial y} = \lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}$$

When you take the partial derivative with respect to x, you treat y as fixed: you treat it as a constant
When you take the partial derivative with respect to y, you treat x as fixed: you treat it as a constant

$$f(x,y) = y^5 - 3xy$$
$$f_x = -3y$$
$$f_y = 5y^4 - 3x$$

$$g(x,y) = \tan^{-1}(xy^2)$$

$$g_x = \frac{1}{1+(xy^2)^2}(y^2)$$

$$g_y = \frac{1}{1+(xy^2)^2}(2xy)$$

Higher Order derivatives do not have to be in the same variable as the first derivative

$$f_{xx} = \frac{\partial^2 f}{\partial x^2}, f_{yy} = \frac{\partial^2 f}{\partial y^2}, f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right] = f_{yx} = \frac{\partial^2 f}{\partial x \partial y}$$

f_{xy}, f_{yx} mixed partials

$$f(x,y) = x^4y^3 + 8x^2y$$

$$f_x = 4x^3y^3 + 16xy$$
$$f_y = 3x^4y^2 + 8x^2$$

$$f_{xx} = 12x^2y^3 + 16y$$
$$f_{yy} = 6x^4y$$

$$f_{xy} = 12x^3y^2 + 16x$$
$$f_{yx} = 12x^3y^2 + 16x$$

$$r(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$$

$$r_u = \langle x_u(u,v), y_u(u,v), z_u(u,v)\rangle$$
$$r_v = \langle x_v(u,v), y_v(u,v), z_v(u,v)\rangle$$

$$r(u,v) = \langle 3\cos(u)\sin(v), 3\sin(u)\sin(v), 3\cos(v)\rangle$$

$$r_u = \langle -3\sin(u)\sin(v)\,, 3\cos(u)\sin(v)\,, 0\rangle$$
$$r_v = \langle 3\cos(u)\cos(v)\,, 3\sin(u)\cos(v)\,, -3\sin\,(v)\rangle$$