5/19/2021

Vector-Valued Functions (13.1)

A vector-valued function is a function of a single variable that is in the form of a vector:

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$

Recall in 2D:

$$r(t) = \langle 3\cos t, 3\sin t \rangle$$

This function is a circle of radius 3.
They are equivalent to a set of parametric equations.

$$r(t) = \langle 3\cos t, \sin t, t \rangle$$

Elliptical helix.
The linear component marks the axis that the helix wraps around.

https://christopherchudzicki.github.io/MathBox-Demos/parametric_curves_3D.html

Domain (and range) of vector-valued functions, and limits

Domain of a vector-value function: find the domain of every component separately, and then take the intersection to be the domain.

$$r(t) = \langle t^3, \ln(3-t), \sqrt{t} \rangle$$

$$x = t^3, y = \ln(3-t), z = \sqrt{t}$$

Find the domain of x, y, and z: x has a domain of all real numbers, y has a domain of $(-\infty, 3)$, and z has a domain of $[0, \infty)$.
The domain of $r(t)$ is $[0,3)$.

Limits of vector-valued functions:
The limit will be a vector, and essentially you find this by finding the limit for each component of the vector.

$$\lim_{t\to c} r(t) = \langle \lim_{t\to c} f(t), \lim_{t\to c} g(t), \lim_{t\to c} h(t) \rangle$$

$$\lim_{t\to 0} \langle 1 + t^3, te^{-t}, \frac{\sin t}{t} \rangle = \langle 1,0,1 \rangle$$

Remind yourself to review parametrization of curves in 2D.

Parametrizing intersections of surfaces:

In general, when dealing with functions: you can replace x with t, and then then y component is just the function in terms of t.

$$x + y = 5$$
$$y + z = 3$$

$$y = t, x = 5 - t, z = 3 - t$$
$$r(t) = \langle 5 - t, t, 3 - t \rangle$$

Suppose I want to parametrize the intersection of the cylinder $x^2 + y^2 = 1$, and $y + z = 2$.

Let $x = \cos t$, $y = \sin t$, substitute y = sint into the equation with z, and solve for z

$$r(t) = \langle \cos t , \sin t , 2 - \sin t \rangle$$

Parametric surfaces (16.6)

Parametrizing 3-variable surfaces (x,y,z) with just two variables (u,v), in a vector
This is a way of working with surfaces that are not functions in (x,y,z), but are functions in (u,v).

$$\vec{r}(u, v) = \langle f(u, v), g(u, v), h(u, v) \rangle$$

A lot of the parametrizations will depend on relating things through cylindrical or spherical coordinates

When using cylindrical coordinates, the radius is typically fixed (so like polar), and then θ is u, and z is v.

When using spherical coordinates, the radius ρ is treated as constant, and the θ and ϕ become u and v.

When you have something which is already a function of (x,y), then just let x=u, and y=v, and z=f(u,v).

Suppose you want to make the parametrization for $f(x, y) = x^2 + 2y^2$.
Let x=u, y=v

$$r(u, v) = \langle u, v, u^2 + 2v^2 \rangle$$

Suppose I wanted to parametrize a sphere: $x^2 + y^2 + z^2 = 16$

$$x = \rho \cos(\theta) \sin(\phi) , y = \rho \sin(\theta) \sin(\phi) , z = \rho \cos (\phi)$$

$$r(u, v) = \langle 4 \cos(u) \sin(v) , 4 \sin u \sin v , 4 \cos v \rangle$$

Suppose we want to parametrize the surface $z = 2\sqrt{x^2 + y^2}$.

$$x = rcos(\theta), y = rsin(\theta), z = z = 2\sqrt{r^2} = 2r$$
$$r = u, \theta = v$$

$$s(u, v) = \langle u \cos v , u \sin v , 2u \rangle$$
$S(u, v) = \langle u, v, 2\sqrt{u^2 + v^2} \rangle$ (works because it's only the top half of the cone)

https://christopherchudzicki.github.io/MathBox-Demos/parametric_surfaces_3D.html

Limits in 2 or more variables (handout on this).

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

Must be path independent.

If the coordinate point (x,y) is within a small distance (δ) of (a,b), then the distance between $f(x,y)$ and L is less than a small value (ε).

(the distance is measured as the Euclidean distance $d = \sqrt{(x-a)^2 + (y-b)^2}$

1. Can you substitute the values of (a,b) into the function…. If the function is defined at that point, you are basically done.
2. If the function is not defined at that point, try simplifying: a) rationalizing, b) factor and cancel, c) doing a substitution (then it may be possible to do L'Hopital's—only works on functions of one variable)
3. May be able to do a change of variables: switch to polar or spherical coordinates

4. Choose a substitution for the "critical path" to test in the equation.

$$\lim_{(x,y)\to(1,2)} (5x^3 - x^2y^2) = 5 - 4 = 1$$

$$\lim_{(x,y)\to(1,0)} \ln\left(\frac{1+y^2}{x^2+xy}\right) = \ln\left(\frac{1}{1}\right) = 0$$

$$\lim_{(x,y)\to(0,0)} \left(\frac{x^4 - 4y^2}{x^2 + 2y^2}\right) = \lim_{y\to 0} -\frac{4y^2}{2y^2} = -2 =? \lim_{x\to 0}\frac{x^4}{x^2} = \lim_{x\to 0} x^2 = 0$$

Could try $x = 0$ or $y = 0$

If two paths produce different values for the limit, then the limit does not exist (DNE)

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^3+y^3} = \lim_{x\to 0}\frac{x^2(kx)}{x^3+k^3x^3} = \lim_{x\to 0}\frac{kx^3}{x^3(1+k^3)} = \frac{k}{1+k^3} = DNE$$

The trick to selecting the "critical path" is $x^3 = y^3 \to x = y$

Substitution I want to make is $x = ky$ or $y = kx$

Another trick to switch to polar

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{r\to 0}\frac{rcos(\theta)rsin(\theta)}{r^2} = \cos(\theta)\sin(\theta) = DNE$$

For three variables, consider switching to spherical

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xy+yz}{x^2+y^2+z^2} = \lim_{\rho\to 0}\frac{\rho\cos(\theta)\sin(\phi)\,\rho\sin(\theta)\sin(\phi) + \rho\sin(\theta)\sin(\phi)\,\rho\cos(\phi)}{\rho^2} =$$

$$\cos(\theta)\sin^2(\phi)\sin(\theta) + \sin(\theta)\sin(\phi)\cos(\phi) = DNE$$

Derivative and Integrals of Vector-Valued Functions

When you take the derivative or an integral of a vector-valued function, you do it component by component.

$$r(t) = \langle \tan t\,, \sec t\,, \frac{1}{t^2} \rangle$$

$$r'(t) = \langle \sec^2 t\,, \sec t \tan t\,, -\frac{2}{t^3} \rangle$$

$$\int r(t)\,dt = \langle -\ln|\cos t| + C_1, \ln|\sec t + \tan t| + C_2, -\frac{1}{t} + C_3 \rangle$$