5/18/2021
Functions of several variables, coordinate systems

Functions of more than one variable (14.1)

An equation in 3 variables, what does that look like? What if it has fewer variables, but is in three dimensions?

In 2D, we consider y to be the dependent variable and x to be the independent variable. So as a function we write $y = f(x)$.

In 3D, we consider z to be dependent variable, and both x and y to be independent variables. So as a function we write $z = f(x, y)$

Domain and range of functions of more than one variable.
The range doesn't change much. The range depends on the possible outcomes of a single variable (z), so we can express it in interval notation just like we did with functions of one variable.

The domain can depend on both x and y, and some relationship between x and y: that can't be written as an interval. Express the domain in set builder notation.

$$D: \{(x, y) | some\ condition\ on\ x\ and\ y\}$$

$$f(x, y) = x^2 + y^2 - 11$$

$$D: \{(x, y) | x \in \mathbb{R}, y \in \mathbb{R}\}$$
$$R: (-11, \infty)$$

$$f(x, y) = \sqrt{x^2 + y^2 - 16}$$

$$x^2 + y^2 - 16 \geq 0$$
$$x^2 + y^2 \geq 16$$

$$D: \{(x, y) | x^2 + y^2 \geq 16\}$$
$$R: [0, \infty)$$

https://c3d.libretexts.org/CalcPlot3D/index.html
GraphCalc (app for computer)

Will circle back to contour curves/level curves and traces

Equations of lines and planes in 3D

Equations of lines are relatively complex in 3D: 1) in parametric form, 2) intersect two planes to obtain the line

Parametric form, or vector form of a line:

$$x = x_0 + at, y = y_0 + bt, z = z_0 + ct$$

$$\vec{r}(t) = \langle x_0 + at, y_0 + bt, z_0 + ct \rangle = \langle x_0, y_0, z_0 \rangle + \langle a, b, c \rangle t$$

As an intersection of planes/symmetric equations

$$\frac{(x - x_0)}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Obtained from the parametric form, by solving for t and equating them, but any two expressions makes a plane, all equal results in the intersection of planes

Find an equation for the line that passes through the points $(-3, -2, 4), (1, -1, 5)$.

$$\langle a, b, c \rangle = \langle -4, -1, -1 \rangle$$

Parametric form:

$$\vec{r}(t) = \langle -3 - 4t, -2 - t, 4 - t \rangle$$

Symmetric form:

$$\frac{x + 3}{-4} = \frac{y + 2}{-1} = \frac{z - 4}{-1}$$

Finding equations of planes in 3D:

A plane in 3D is defined by a point in the plane and the vector perpendicular to the surface ($\vec{n} = \langle a, b, c \rangle$).

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$ax + by + cz + d = 0$$

We want to find an equation of the plane containing the points $(1, -1, 2), (2, -4, 3), (3, 5, 2)$.

First: find two vectors in the plane

$$\vec{v}_1 = \langle -1, 3, -1 \rangle, \vec{v}_2 = \langle -2, -6, 0 \rangle$$

Second: find a vector perpendicular to the plane by doing the cross product of these two vectors

$$\vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} i & j & k \\ -1 & 3 & -1 \\ -2 & -6 & 0 \end{vmatrix} = (0 - 6)i - (0 - 2)j + (6 + 6)k = \langle -6, 2, 12 \rangle$$

Write equation: $-6(x - 1) + 2(y + 1) + 12(z - 2) = 0$

Distance between a point and a line, or a point and a plane; and can be useful for determining the angle between two planes

If the normal vector that defines two planes are parallel (multiples of each other), then the planes are also parallel (or identical).

If planes intersect, we can find the angle between the plane: by finding the angle between the normal vectors defining the plane.

$$x + y + z = 3$$
$$x - 2y + 3z = 10$$

$$\vec{n}_1 = \langle 1,1,1 \rangle, \vec{n}_2 = \langle 1,-2,3 \rangle$$

$$\cos(\theta) = \frac{1 - 2 + 3}{\sqrt{3}\sqrt{14}} = \frac{2}{\sqrt{42}}$$
$$\cos^{-1}\left(\frac{2}{\sqrt{42}}\right) = 72.02\ degrees, or\ 1.257\ radians$$

Distance between a plane and a line:

The distance between a point and plane: a point in space (P), a point in the plane (Q), and the normal vector to the plane (n)

The distance between a point a line: a point in space (P), a point on the line (Q), and the vector the defines the direction of the line (v).

$$d = \frac{|\overrightarrow{QP} \cdot n|}{|n|}$$

$$d = \frac{|\overrightarrow{QP} \times v|}{|v|}$$

Find the distance between the point and the plane: $(1, 2{,}3), x + y + z = 3$

$$P = (1,2,3), Q = (3,-1,1), n = \langle 1,1,1 \rangle$$
$$PQ = \langle -2,3,2 \rangle$$

$$PQ \cdot n = -2 + 3 + 2 = 3$$
$$|n| = \sqrt{3}$$
$$d = \frac{3}{\sqrt{3}}$$

Find the distance between point (1,2,3) and the line $\frac{x-2}{3} = \frac{y+1}{2} = z - 4$

$$Q = (2,-1,4), v = \langle 3,2,1 \rangle$$

$$PQ = \langle -1,3,-1 \rangle$$

$$PQ \times v = \begin{vmatrix} i & j & k \\ -1 & 3 & -1 \\ 3 & 2 & 1 \end{vmatrix} = (3+2)i - (-1+3)j + (-2-9)k = \langle 5, -2, -11 \rangle$$

$$d = \frac{\sqrt{25+4+121}}{\sqrt{9+4+1}} = \frac{\sqrt{150}}{\sqrt{14}} = \frac{\sqrt{75}}{\sqrt{7}}$$

Cylinders and Quadric Surfaces

Circular cylinder is $x^2 + y^2 = 1$ (cylinder wrapped the axis of the missing variable, z-axis)
$x^2 + z^2 = 1$ (wrapped around the y-axis)
$y^2 + z^2 = 1$ (wrapped around the x-axis)

Quadric surfaces are the 3D equivalent of quadratic curves in 2D
$y = x^2$ or $x = y^2$ parabolas
$ax^2 + by^2 = c$ circles/ellipses
$ax^2 - by^2 = c$ hyperbolas

$$ax^2 + by^2 + cz^2 = d$$

Ellipsoid (sphere if a=b=c)

Paraboloid $z = ax^2 + by^2$ (one linear term terms the axis the shape is wrapped around)

Hyperbolic paraboloid $z = ax^2 - by^2$ (one linear term + change in sign of squared terms, linear term determines orientation)

Hyperboloid: 1) of 1 sheet, vs. 2) of two sheets (determined by the number of negative signs)

$ax^2 + by^2 - cz^2 = d$ (hyperboloid of one sheet wrapped about the z-axis, by where the negative sign is)

$ax^2 - by^2 - cz^2 = d$ (hyperboloid of two sheets, wrapped around the x-axis, determined by where the positive sign)

of d goes to 0, then you get a cone:

$ax^2 + by^2 = cz^2$ is a cone (the variable on one side by itself (or a different sign), is the axis it's wrapped around).

Coordinate systems:

Polar Coordinates:

$$x^2 + y^2 = r^2$$
$$x = rcos(\theta)$$
$$y = rsin(\theta)$$
$$\tan^{-1}\left(\frac{y}{x}\right) = \theta$$

Cylindrical coordinates = polar coordinates + z

Conversion formulas a exactly the same, just add $z = z$.

Suppose you have a point (1,0,4) in rectangular coordinates, and you want to convert to cylindrical coordinates.

Convert (1,0) to polar, and then tack on the z.

$x = 1, y = 0, r = 1, \theta = \tan^{-1}\left(\frac{0}{1}\right) = 0$

$$(r, \theta) = (1,0)$$

In cylindrical: $(1,0,4) = (1,0,4) = (r, \theta, z)$

Spherical

ρ is the distance (straight-line distance) from the origin
θ is the angle from the positive x-axis in the xy-plane (longitude)
ϕ is the angle from the positive z-axis (similar to latitude)

$$x^2 + y^2 + z^2 = \rho^2$$
$$x = \rho \cos(\theta) \sin(\phi)$$
$$y = \rho \sin(\theta) \sin(\phi)$$
$$z = \rho \cos(\phi)$$

$$\tan^{-1}\left(\frac{y}{x}\right) = \theta$$
$$\phi = \cos^{-1}\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right)$$

$$r^2 = x^2 + y^2 = \rho^2 \sin^2 \phi$$
$$r = \rho \sin(\phi)$$

$$(\rho, \theta, \phi)$$

$$x^2 + y^2 = z$$

Convert to cylindrical (z is function variable)

$$r^2 = z$$

Convert to spherical (ρ is the function variable)

$$\rho^2 \sin^2 \phi = \rho \cos \phi$$

$$\rho = \cot \phi \csc \phi$$

In Stewart book (15.8, 15.9)