MTH 265, Exam #1, Summer 2021 Name KEY

Instructions: Show all work. Use exact answers unless specifically asked to round. You may check your answers in the calculator, but you must show work to get full credit. Incorrect answers with no work will receive no credit. Be sure to complete all the requested elements of each problem.

1. Find $\vec{u} \times \vec{v}$ if $\vec{u} = \langle 1,2,3 \rangle$ and $\vec{v} = \langle 4,-9,0 \rangle$. (8 points)

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 4 & -9 & 0 \end{vmatrix} = (0+27)\hat{i} - (0-12)\hat{j} + (-9-8)\hat{k}$$

$$\langle 27, 12, -17 \rangle$$

2. Calculate the volume of the parallelepiped defined by the vectors $\langle 1,2,0 \rangle, \langle 3,4,6 \rangle, \langle 1,1,2 \rangle$. (9 points)

$u \quad v \quad w$

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & 4 & 6 \\ 1 & 1 & 2 \end{vmatrix} = 1(8-6) - \cancel{2(6-6)}^{0} + \cancel{0(3-4)}^{0} = 2$$

3. Find the equation of the plane containing the points $\overset{A}{(2,1,1)}$, $\overset{B}{(1,2,11)}$, and $\overset{C}{(3,2,4)}$. (12 points)

$$\vec{AB} = \langle -1, 1, 10 \rangle \qquad \vec{BC} = \langle 2, 0, -7 \rangle$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & 10 \\ 2 & 0 & -7 \end{vmatrix} = (-7-0)\hat{i} - (7-20)\hat{j} + (0-2)\hat{k} = \langle -7, 13, -2 \rangle = \vec{n}$$

$$-7(x-2) + 13(y-1) - 2(z-1) = 0$$

4. Find the symmetric form of the line perpendicular to the plane $2x - 3y + z = 13$ and passes through the point (0,1,5). (8 points)

$\langle 2, -3, 1 \rangle$

$$\frac{x}{2} = \frac{y-1}{-3} = \frac{z-5}{1}$$

5. Rewrite $z = 2 - \sqrt{x^2 + y^2}$ in cylindrical and spherical coordinates. Simplify as much as possible. (8 points)

$z = 2 - r$ cylindrical

$\rho \cos\varphi = 2 - \rho \sin\varphi$

$\rho \cos\varphi + \rho \sin\varphi = 2$

$\rho = \dfrac{2}{\cos\varphi + \sin\varphi}$ spherical

6. Find the limit of the following expressions at the indicated point. If the limit exists, state its value. If the limit does not exist, explain why not. You will need to test multiple paths. You may wish to use polar or spherical coordinates. (7 points each)

a. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^3 + y^2}$ $x^3 = y^2 \Rightarrow x^{3/2} = y$ let $y = kx^{3/2}$

$$\lim_{x\to 0} \frac{x^2 k x^{3/2}}{x^3 + (kx^{3/2})^2} = \lim_{x\to 0} \frac{kx^{7/2}}{x^3(1+k^2)} = \lim_{x\to 0} \frac{kx^{1/2}}{1+k^2} = 0$$

b. $\lim_{(x,y)\to(0,0)} \frac{\sin\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}$

$x^2+y^2=r^2$

$\lim_{r\to 0} \frac{\sin r}{r} = 1$

c. $\lim_{(x,y,z)\to(0,0,0)} \frac{2x^2+yz+4xz}{x^2+y^2+z^2}$

$\lim_{\rho\to 0} \frac{2\cancel{\rho^2}\cos^2\theta\sin^2\varphi + \cancel{\rho^2}\sin\theta\sin\varphi\cos\varphi + 4\cancel{\rho^2}\cos\theta\sin\varphi\cos\varphi}{\cancel{\rho^2}}$

$= 2\cos^2\theta\sin^2\varphi + \sin\theta\sin\varphi\cos\varphi + \cos\theta\sin\varphi\cos\varphi$

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d. $\lim_{(x,y)\to(0,0)} \frac{x-y}{\sqrt{x}-\sqrt{y}}$ $\frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}+\sqrt{y}}$

$\lim_{(x,y)\to(0,0)} \frac{(\sqrt{x}+\sqrt{y})\cancel{(x-y)}}{\cancel{(x-y)}} = 0$

7. Find $\vec{r}'(t)$ for vector of the vector-valued function $\vec{r}(t) = cos^2 t\hat{\imath} + sin^2 t\hat{\jmath}$. Find an expression for the length of the vector. (8 points)

$r'(t) = -2\cos t\sin t\,\hat{\imath} + 2\sin t\cos t\,\hat{\jmath}$

$\|r'(t)\| = \sqrt{4\cos^2 t\sin^2 t + 4\sin^2 t\cos^2 t} = \sqrt{8\cos^2 t\sin^2 t}$

$= 2\sqrt{2}\cos t\sin t$

8. Integrate the vector-valued function $\vec{r}(t) = \langle t, 2t^2, t^3 \rangle$ on the interval [0,2]. (7 points)

$$\int_0^2 r(t)\,dt = \int_0^2 t\,dt\,\hat{i} + \int_0^2 2t^2\,dt\,\hat{j} + \int_0^2 t^3\,dt\,\hat{k}$$

$$\frac{1}{2}t^2\Big|_0^2 \hat{i} + \frac{2}{3}t^3\Big|_0^2 \hat{j} + \frac{t^4}{4}\Big|_0^2 \hat{k} = 2\hat{i} + \frac{16}{3}\hat{j} + 4\hat{k}$$

$$\langle 2, \frac{16}{3}, 4 \rangle$$

9. Find the value of the line integral $\int_C (x+2y)dx + (3x^2 - 4y)dy$ on the curve $\vec{r}(t) = 2t\hat{\imath} - t^2\hat{\jmath}$ over the interval [0,3]. (10 points)

$dx = 2dt$
$dy = -2t\,dt$

$$\int_0^3 (2t - 2t^2)\,2\,dt + (3 \cdot 4t^2 + 4t^2)(-2t\,dt)$$

$$= \int_0^3 4t - 4t^2 - 24t^3 - 8t^3\,dt = \int_0^3 4t - 4t^2 - 32t^3\,dt$$

$$2t^2 - \frac{4}{3}t^3 - 8t^4\Big|_0^3 = 18 - 36 - 648 = -666$$

10. Find the value of the line integral $\int_C x^2 yz\,ds$ over the line segment connecting the points (1,2,-1) and (2,-1,3). (16 points)

$\vec{u} = \langle 1, -3, 4 \rangle$
$r(t) = \langle 1+t, 2-3t, -1+4t \rangle$
$r'(t) = \langle 1, -3, 4 \rangle$
$\|r'(t)\| = \sqrt{1+9+16} = \sqrt{26}$

$$\int_0^1 (1+t)^2(2-3t)(-1+4t)\sqrt{26}\,dt$$

$$\int_0^1 (1+2t+t^2)(-2-4t-12t^2)\sqrt{26}\,dt$$

$$\sqrt{26}\int_0^1 -2 - 4t - 12t^2 - 4t - 8t^2 - 24t^3 - 2t^2 - 4t^3 - 12t^4\,dt$$

$$\sqrt{26}\int_0^1 -2 - 8t - 22t^2 - 28t^3 - 12t^4\,dt =$$

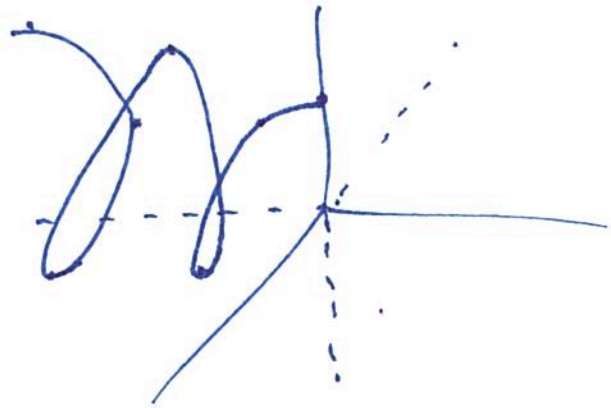
$$\sqrt{26}\left[-2t - 4t^2 - \frac{22}{3}t^3 - \frac{28}{4}t^4 - \frac{12}{5}t^5\Big|_0^1 = \sqrt{26}\left[-2 - 4 - \frac{22}{3} - \frac{28}{4} - \frac{12}{5}\right] =\right.$$

$-341/15 =$

$-22.7\overline{3}$

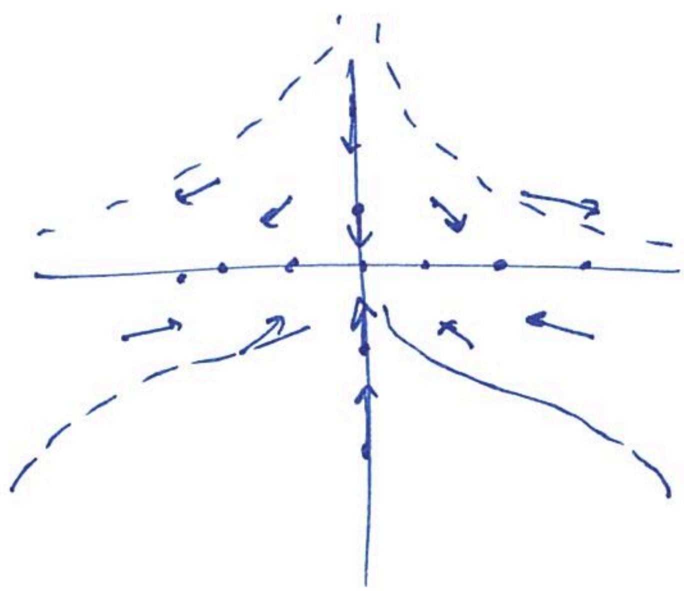
11. Sketch the graph of the vector valued function $\vec{r}(t) = \sin 2t\,\hat{\imath} - 3t\hat{\jmath} + \cos 2t\,\hat{k}$. Plot at least 9 points. (10 points)

2t	x	y	z
0	0	0	1
$\pi/2$	1	$-3\pi/4$	0
π	0	$-3\pi/2$	-1
$3\pi/2$	-1	$-9\pi/4$ -3π	0
2π	0	$-15\pi/4$	1
$5\pi/2$	1	$-9\pi/2$	0
3π	0		-1
$7\pi/2$	-1	$-21\pi/4$	0
4π	0	-6π	1



12. Sketch the vector field $\vec{F}(x, y) = xy\hat{\imath} - y\hat{\jmath}$. Plot at least 10 vectors (or more: whatever is required to determine the behaviour of the field). (12 points)

x	y	$\langle xy, -y \rangle$
0	0	$\langle 0,0 \rangle$
0	1	$\langle 0,-1 \rangle$
1	0	$\langle 0,0 \rangle$
0	-1	$\langle 0,1 \rangle$
-1	0	$\langle 0,0 \rangle$
1	1	$\langle 1,-1 \rangle$
-1	1	$\langle -1,-1 \rangle$
-1	-1	$\langle 1,1 \rangle$
1	-1	$\langle -1,1 \rangle$
2	1	$\langle 2,-1 \rangle$
2	0	$\langle 0,0 \rangle$



13. Find a parametric expression in two variables for the surface $z = \sqrt{x^2 + y^2}$. (7 points)

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$z = v$

$x = v\cos u$

$y = v\sin u$

$$r(u, v) = v\cos u\,\hat{i} + v\sin u\,\hat{j} + v\hat{k}$$

$v \geq 0$

$0 \leq u \leq 2\pi$