Variation of Parameters. Solution.

Directions: Do one step, and then pass it along to the next student. You do not have to solve the entire problem. If you see a mistake, correct it. If you are not sure, discuss. I will check back.

Solve the problem below.

1. $y'' + 2y' + y = x^2 + e^{-x}$

Since the characteristic equation is $k^2 + 2k + 1 = 0$ and the roots are $(k+1)^2 = 0, k = -1$ (repeated), the homogeneous solution is $y = c_1 e^{-x} + c_2 x e^{-x}$. The Wronksian is

$$W = \begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & e^{-x} - xe^{-x} \end{vmatrix} = e^{-2x} - xe^{-2x} + xe^{-2x} = e^{-2x}$$

The variation of parameters formula, in terms of just y's and x's (and not u's) is

$$y_p = -y_1 \int \frac{y_2 g(x)}{W} dx + y_2 \int \frac{y_1 g(x)}{W} dx$$

So, we have

$$y_p = -e^{-x} \int \frac{xe^{-x}(x^2 + e^{-x})}{e^{-2x}} dx + xe^{-x} \int \frac{e^{-x}(x^2 + e^{-x})}{e^{-2x}} dx =$$

$$-e^{-x} \int \frac{x^3 e^{-x} + xe^{-2x}}{e^{-2x}} dx + xe^{-x} \int \frac{x^2 e^{-x} + e^{-2x}}{e^{-2x}} dx =$$

$$-e^{-x} \int (x^3 e^x + x) dx + xe^{-x} \int (x^2 e^x + 1) dx =$$

$$-e^{-x} \left[x^3 e^x - 3x^2 e^x + 6xe^x - 6e^x + \frac{1}{2}x^2 \right] + xe^{-x}[x^2 e^x - 2xe^x + 2e^x + x] =$$

$$-x^3 + 3x^2 - 6x + 6 - \frac{1}{2}x^2 e^{-x} + x^3 - 2x^2 + 2x + x^2 e^{-x} =$$

$$x^2 - 4x + 6 + \frac{1}{2}x^2 e^{-x}$$

Thus, the final solution is

$$\boldsymbol{y = c_1 e^{-x} + c_2 x e^{-x} + x^2 - 4x + 6 + \frac{1}{2}x^2 e^{-x}}$$