Undetermined Coefficients. Solutions.

Directions: Do one step, and then pass it along to the next student. You do not have to solve the entire problem. If you see a mistake, correct it. If you are not sure, discuss. I will check back.

Solve the problem below.

1. $y'' - 5y' - 24y = 3\sin x + 2\cos 7x$

Since the characteristic equation is $k^2 - 5k - 24 = 0$, the solution to the homogeneous problem is $y_h = c_1 e^{8x} + c_2 e^{-3x}$. This does not match the forcing function, so our Ansatz is $y_p = A\sin x + B\cos x + C\sin 7x + D\cos 7x$.

$$y_p' = A\cos x - B\sin x + 7C\cos x - 7D\sin 7x,$$
$$y_p'' = -A\sin x - B\cos x - 49\sin 7x - 49\cos 7x$$

$$-A\sin x - B\cos x - 49\sin 7x - 49\cos 7x - 5A\cos x + 5B\sin x - 35C\cos x + 35D\sin 7x - 24A\sin x - 24B\cos x - 24C\sin 7x - 24D\cos 7x = 3\sin x + 2\cos 7x$$

$$\sin x: -25A + 5B = 3$$
$$\cos x: -5A - 25B = 0$$
$$\sin 7x: -73C + 35D = 0$$
$$\cos 7x: -35C - 73D = 2$$

$$A = -\frac{3}{26}, B = \frac{3}{130}, C = -\frac{35}{3277}, D = -\frac{73}{3277}$$

So, $\boldsymbol{y_p = -\frac{3}{26}\sin x + \frac{3}{130}\cos x - \frac{35}{3277}\sin 7x - \frac{73}{3277}\cos 7x}$.

Thus $\boldsymbol{y = c_1 e^{8x} + c_2 e^{-3x} - \frac{3}{26}\sin x + \frac{3}{130}\cos x - \frac{35}{3277}\sin 7x - \frac{73}{3277}\cos 7x}$

2. $y'' + 2y' + 5y = x^2 + e^{-x}$

The homogeneous solution is $y_h = c_1 e^{-x}\sin 2x + c_2 e^{-x}\cos 2x$. The forcing function is not one of these terms, so your Ansatz is $y_p = Ax^2 + Bx + C + De^{-x}$. $y_p' = 2Ax + B - De^{-x}$, $y_p'' = 2A + De^{-x}$.

$$2A + De^{-x} + 4Ax + 2B - 2De^{-x} + 5Ax^2 + 5Bx + 5C + 5De^{-x} = x^2 + e^{-x}$$

$$x^2: 5A = 1$$
$$x: 4A + 5B = 0$$
$$1: 2A + 2B + 5C = 0$$
$$e^{-x}: 4D = 1$$
$$A = \frac{1}{5}, B = -\frac{4}{25}, C = -\frac{2}{125}, D = \frac{1}{4}$$

So, $y_p = \frac{1}{5}x^2 - \frac{4}{25}x - \frac{2}{125} + \frac{1}{4}e^{-x}$.
And thus $\boldsymbol{y = c_1 e^{-x} \sin 2x + c_2 e^{-x} \cos 2x + \frac{1}{5}x^2 - \frac{4}{25}x - \frac{2}{125} + \frac{1}{4}e^{-x}}$.

3. $y'' + 8y' = x - \cos x$

Since the solutions to the homogeneous equation are $k = 0, k = -8$, the homogeneous solution is $y_h = c_1 + c_2 e^{-8x}$. Since our initial Ansatz for x is $Ax + B$, and the B is a copy of the homogeneous solution, multiply the guess by x. Following that, the Ansatz becomes $y_p = Ax^2 + Bx + C \sin x + D \cos x$. $y_p' = 2Ax + B + C \cos x - D \sin x$, $y_p'' = 2A - C \sin x - D \cos x$.

$$2A - C \sin x - D \cos x + 2Ax + B + C \cos x - D \sin x = x - \cos x$$

$$x^2: 0 = 0$$
$$x: 2A = 1$$
$$1: 2A + B = 0$$
$$\sin x: -C - D = 0$$
$$\cos x: C - D = -1$$

$$A = \frac{1}{2}, B = -1, C = -\frac{1}{2}, D = \frac{1}{2}$$

So, the particular solution is $y_p = \frac{1}{2}x^2 - x - \frac{1}{2}\sin x + \frac{1}{2}\cos x$, making the solution $\boldsymbol{y = c_1 + c_2 e^{-8x} + \frac{1}{2}x^2 - x - \frac{1}{2}\sin x + \frac{1}{2}\cos x}$.