Springs, solutions.

Directions: Do one step, and then pass it along to the next student. You do not have to solve the entire problem. If you see a mistake, correct it. If you are not sure, discuss. I will check back.

Solve the problem below.

1. A mass weighing 6 pounds is attached to a spring and stretches it 4 inches. The spring is stretched an additional 6 inches from equilibrium and released with an upward velocity of 1 inch per second. A forcing function of $F(x) = \sin x$ is applied to the system.
   a. Set up the equation of the system, including stating any initial conditions.
   b. Solve the system.
   c. Does the system exhibit resonance or beats?
   d. Graph the system.
   e. Describe the long-term behavior of the system.

$$F = ky \to 6 = k\left(\frac{1}{3}\right) \to k = 18$$

$$F = ma \to 6 = m(32) \to m = \frac{3}{16}$$

$$\frac{3}{16}y'' + 18y = \sin x$$

$$y'' + 96y = \frac{16}{3}\sin x\,, y(0) = -\frac{1}{2}, y'(0) = \frac{1}{12}$$

$$k^2 + 96 = 0 \to k = \pm 9.798i$$

$$y_h = c_1 \sin 9.798x + c_2 \cos 9.798x\,, y_p = A\sin x + B\cos x$$

$$-A\sin x - B\cos x + 96A\sin x + 96B\cos x = \frac{16}{3}\sin x$$

$$95A\sin x + 95B\cos x = \frac{16}{3}\sin x$$

$$A = \frac{16}{285}, B = 0$$

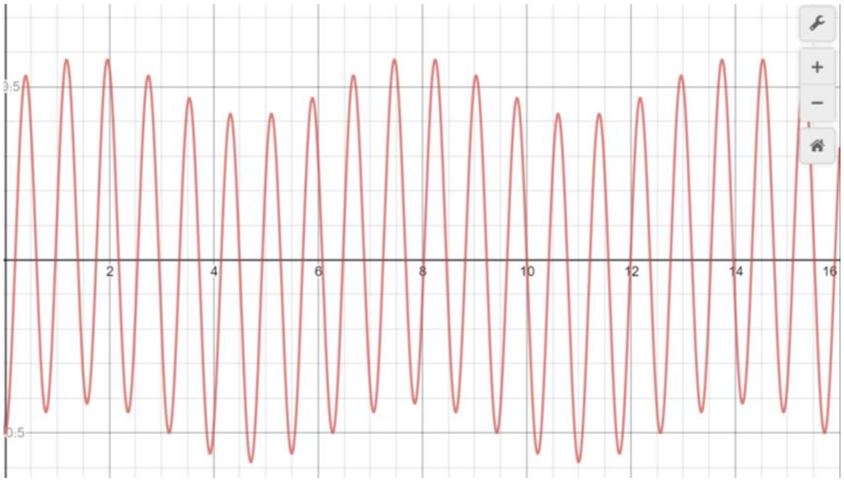
$$y(x) = c_1 \sin 9.798x + c_2 \cos 9.798x + \frac{16}{285}\sin x$$

$$y'(x) = 9.798c_1 \cos 9.798x - 9.798c_2 \sin 9.798x + \frac{16}{285}\cos x$$

$$\frac{1}{12} = 9.798c_1 - 0 + \frac{16}{285} \to 9.798c_1 = -\frac{1}{756} \to c_1 = -1.35 \times 10^{-4}$$

$$-\frac{1}{2} = 0 + c_2 + 0 \to c_2 = -\frac{1}{2}$$

$$y(x) = -1.35 \times 10^{-4} \sin 9.798x - \frac{1}{2}\cos 9.798x + \frac{16}{285}\sin x$$



Most similar to beats, but frequencies are not very similar. The long-term behavior persists like this since there is no damping.

2. A mass weighing 19 kg is attached to a spring and stretches it 13 centimeters. The spring is attached to a damping mechanism that applies a damping force at twice the magnitude of the velocity. The spring is pushed upward from equilibrium by 5 cm and released. A forcing function of $F(x) = e^{-x} + \cos 2x$ is applied to the system. You may round components of your solution to 4 decimal places (they will be pretty ugly).
   a. Set up the equation of the system, including stating any initial conditions.
   b. Solve the system.
   c. Is the system undamped, underdamped, critically damped or overdamped?
   d. Graph the system.
   e. Which parts of the equation are transient, and which are steady state?

$$m = 19 \to F = 19 \times 9.8 = 186.2N$$

$$F = ky \to 186.2 = k(0.13) \to k = \frac{18620}{13} \approx 1432.3077$$

$$\gamma = 2, y(0) = 0.05, y'(0) = 0$$

$$19y'' + 2y' + 1432.3077y = e^{-x} + \cos 2x$$

$$19k^2 + 2k + 1432.3077 = 0$$

$$k = \frac{\left(-2 \pm \sqrt{4 - 4(19)(1432.3077)}\right)}{2(19)} = \frac{-2 \pm \sqrt{-108551.384}}{38} = \frac{-2 \pm 329.9263i}{38}$$
$$= -\frac{1}{19} \pm 8.6823i$$

$$y_h = c_1 e^{-\frac{1}{19}x} \sin 8.6823x + c_2 e^{-\frac{1}{19}x} \cos 8.6823x$$

$$y_p = Ae^{-x} + B \sin 2x + C \cos 2x$$

$$y_p' = -Ae^{-x} + 2B\cos 2x - 2C\sin 2x$$
$$y_p'' = Ae^{-x} - 4B\sin 2x - 4C\cos 2x$$

$$19[Ae^{-x} - 4B\sin 2x - 4C\cos 2x] + 2[-Ae^{-x} + 2B\cos 2x - 2C\sin 2x] + 1432.3077[Ae^{-x} + B\sin 2x + C\cos 2x] = e^{-x} + \cos 2x$$

$$e^{-x}: 19A - 2A + 1432.3077A = 1 \rightarrow 1449.3077A = 1 \rightarrow A = 6.9 \times 10^{-4}$$

$$\sin 2x: -76B - 4C + 1432.3077B = 0 \rightarrow 1356.3077B - 4C = 0$$
$$\cos 2x: -76C + 4B + 1432.3077C = 1 \rightarrow 4B + 1356.3077C = 1$$

$$B = 2.1744 \times 10^{-6}, C = 7.3729 \times 10^{-4}$$

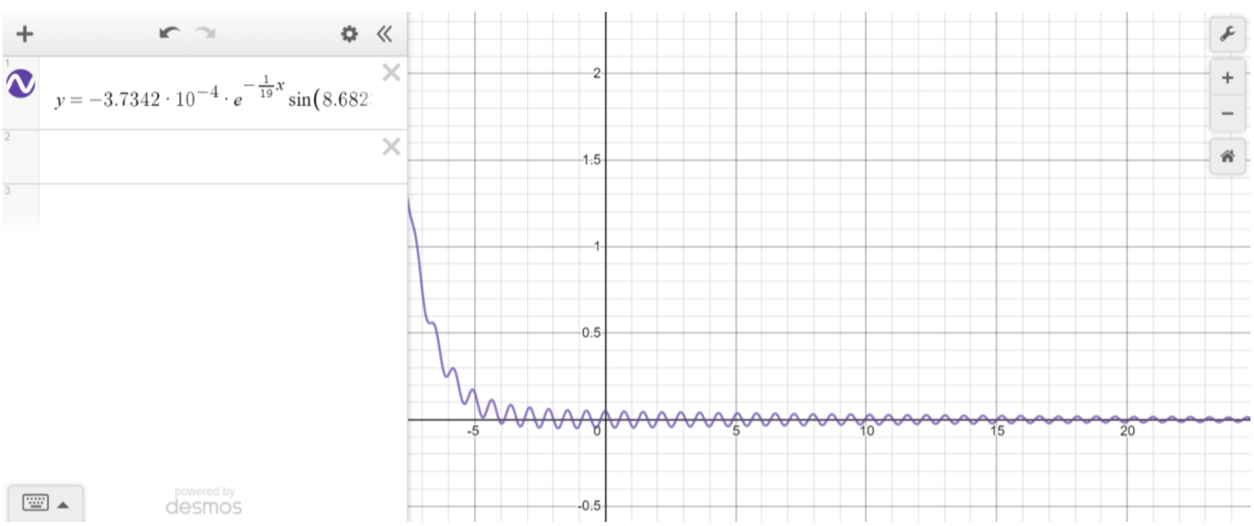
$$y(x) = c_1 e^{-\frac{1}{19}x}\sin 8.6823x + c_2 e^{-\frac{1}{19}x}\cos 8.6823x + 6.9 \times 10^{-4} e^{-x} + 2.1744 \times 10^{-6}\sin 2x + 7.3729 \times 10^{-4}\cos 2x$$

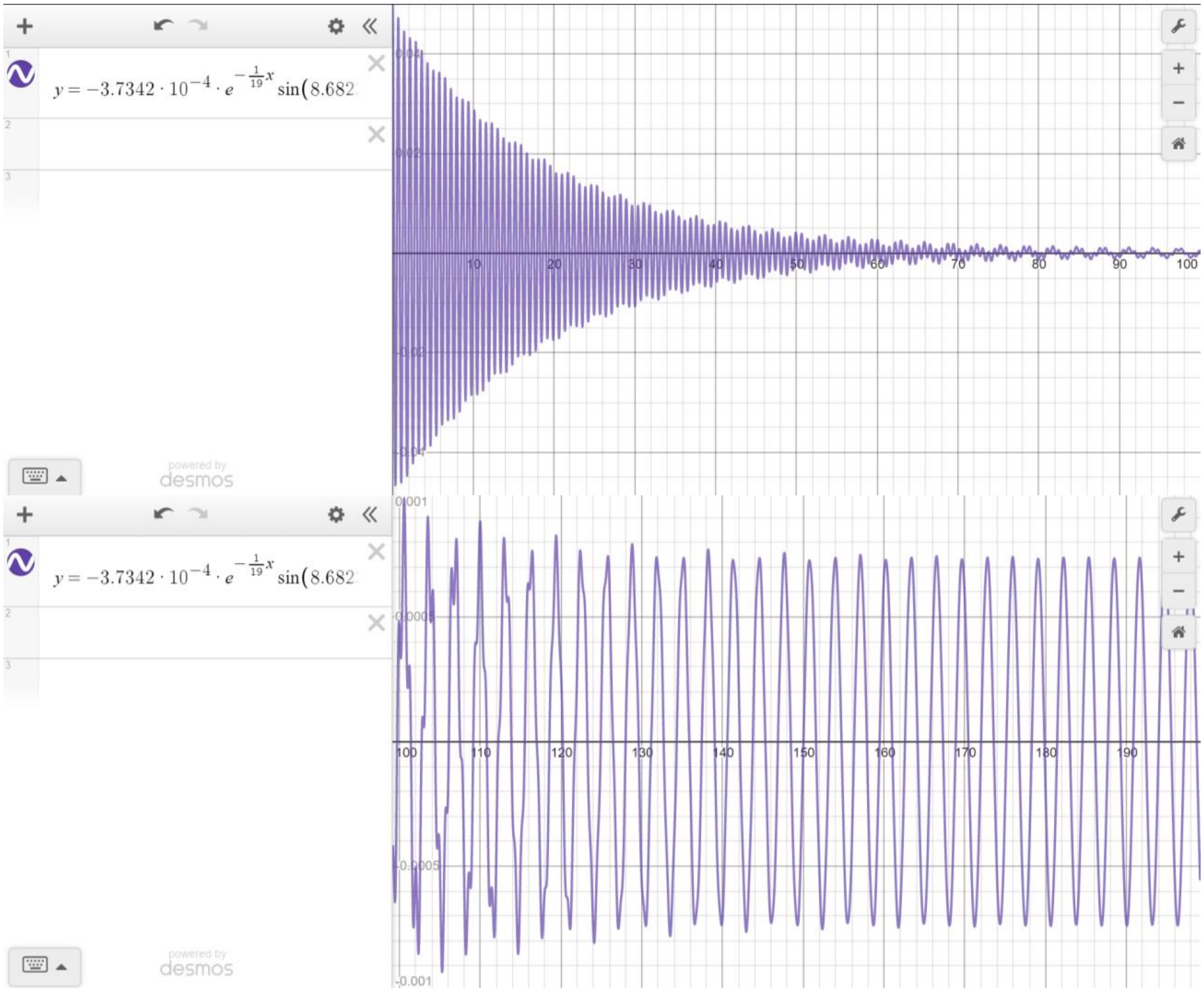
$$y' = -\frac{1}{19}c_1 e^{-\frac{1}{19}x}\sin 8.6823x + 8.6823c_1 e^{-\frac{1}{19}x}\cos 8.6823x - \frac{1}{19}c_2 e^{-\frac{1}{19}x}\cos 8.6823x - 8.6823c_2 e^{-\frac{1}{19}x}\sin 8.6823x - 6.9 \times 10^{-4}e^{-x} + 4.3488 \times 10^{-6}\cos 2x + 0.001475\sin 2x$$

$$0.05 = 0 + c_2 + 6.9 \times 10^{-4} + 0 + 7.3729 \times 10^{-4} \rightarrow c_2 = 0.04857$$

$$0 = 0 + 8.6823c_1 - \frac{1}{19}c_2 - 0 - 6.9 \times 10^{-4} + 4.3488 \times 10^{-6} + 0 \rightarrow c_1 = -3.7342 \times 10^{-4}$$

$$y(x) = -3.7342 \times 10^{-4} e^{-\frac{1}{19}x}\sin 8.6823x + 0.04857e^{-\frac{1}{19}x}\cos 8.6823x + 6.9 \times 10^{-4}e^{-x} + 2.1744 \times 10^{-6}\sin 2x + 7.3729 \times 10^{-4}\cos 2x$$





This function is underdamped.
All the exponential terms are decaying, so they are transient (though, you can see from the graph, they do take a while to decay), the remaining terms that are pure sine and cosine are the steady state components.