Reduction of Order. Solutions.

Directions: Do one step, and then pass it along to the next student. You do not have to solve the entire problem. If you see a mistake, correct it. If you are not sure, discuss. I will check back.

Solve the problem below.

1. $xy'' - y' + (1-x)y = 0, \ y_1 = e^x$

$$y_2 = ve^x, y' = v'e^x + ve^x, y'' = v''e^x + 2v'e^x + ve^x$$

$$x(v''e^x + 2v'e^x + ve^x) - (v'e^x + ve^x) + (1-x)ve^x = 0$$

$$v''xe^x + 2v'xe^x + vxe^x - v'e^x - ve^x + ve^x - vxe^x = 0$$

$$v''xe^x + v'(2xe^x - e^x) = 0, let\ v' = u, v'' = u'$$

$$u'xe^x = u(e^x - 2xe^x)$$

$$\frac{du}{u} = \frac{e^x - 2xe^x}{xe^x} dx = \frac{1}{x} - 2\,dx$$

$$\ln u = \ln x - 2x = \ln(xe^{-2x}) \to u = xe^{-2x} \to v = \int xe^{-2x} dx = -\frac{1}{4} e^{-2x}(2x+1)$$

$$y_2 = e^x\left(e^{-2x}(2x+1)\right) = e^{-x}(2x+1)$$

$$y_h = c_1 e^x + c_2(2x+1)e^{-x}$$

2. Find the Wronskian for the fundamental set above.

$$W = \begin{vmatrix} e^x & (2x+1)e^{-x} \\ e^x & 2e^{-x} - (2x+1)e^{-x} \end{vmatrix} = 2 - 2x - 1 - 2x - 1 = -4x$$

3. Use that information to solve $xy'' - y' + (1-x)y = 4x^2e^x$ using any appropriate method.

$$y_p = -e^x \int \frac{(2x+1)e^{-x}4x^2e^x}{-4x} dx + (2x+1)e^{-x} \int \frac{e^x 4x^2 e^x}{-4x} dx =$$

$$-e^x \int -x dx + (2x+1)e^{-x} \int -xe^{2x} dx =$$

$$-e^x\left(-\frac{x^2}{2}\right) + (2x+1)e^{-x}\left(\frac{1}{4}\right)e^{2x}(1-2x) = \frac{e^x x^2}{2} + \frac{1}{4}(1-4x^2)e^x = -\frac{1}{4}e^x(2x^2-1)$$

$$y(x) = c_1 e^x + c_2(2x+1)e^{-x} - \frac{1}{4}e^x(2x^2 - 1)$$