MTH 291, Practice Final Exam, Summer 2018 Name ____

Instructions: Show all work. Answers without work required to obtain the solution will not receive full credit. Some questions may contain multiple parts: be sure to answer all of them. Give exact answers unless specifically asked to estimate.

1. Estimate the solution of the ODE $\frac{dy}{dx} = y - xy, y(0) = 2$ using $\Delta t = 0.1$ using two complete steps of Runge-Kutta.

2. Rewrite $y'' + y = \cos 2t - 6 \sin 2t$ as a system of first order equations. (You don't need to solve.)

3. Solve $\vec{x}' = \begin{pmatrix} 1 & 3 \\ -1 & -2 \end{pmatrix} \vec{x}$. Write the general solution (with real terms only). Plot several sample trajectories.

4. Verify that $\Psi = \begin{pmatrix} 3e^{-2t} & e^t & e^{3t} \\ -2e^{-2t} & -e^t & -e^{3t} \\ 2e^{-2t} & e^t & 0 \end{pmatrix}$ is a solution to $\vec{x}' = \begin{pmatrix} -8 & -11 & -2 \\ 6 & 9 & 2 \\ -6 & -6 & 1 \end{pmatrix} \vec{x}$. Does the solution represent a fundamental set?

5. Solve $\vec{x}' = \begin{bmatrix} 1 & -3 \\ 3 & 7 \end{bmatrix} \vec{x}$ for the general solution (with real terms only). Describe the behavior of the origin: a repeller, attracter, or saddle point.

6. The fundamental solution matrix to the system $\vec{x}' = \begin{pmatrix} 2 & 4 \\ 2 & -5 \end{pmatrix} \vec{x}$ is $\Psi = \begin{pmatrix} -e^{-6t} & -4e^{3t} \\ 2e^{-6t} & e^{3t} \end{pmatrix}$. Use this fact to solve the system $\vec{x}' = \begin{pmatrix} 2 & 4 \\ 2 & -5 \end{pmatrix} \vec{x} + \begin{pmatrix} 3e^t \\ -t^2 \end{pmatrix}$ with variation of parameters.

7. The fundamental solution matrix to the system $\vec{x}' = \begin{pmatrix} 2 & 4 \\ 2 & -5 \end{pmatrix} \vec{x}$ is $\Psi = \begin{pmatrix} -e^{-6t} & -4e^{3t} \\ 2e^{-6t} & e^{3t} \end{pmatrix}$. Use this fact to solve the system $\vec{x}' = \begin{pmatrix} 2 & 4 \\ 2 & -5 \end{pmatrix} \vec{x} + \begin{pmatrix} 3e^t \\ -t^2 \end{pmatrix}$ using the method of undetermined coefficients.

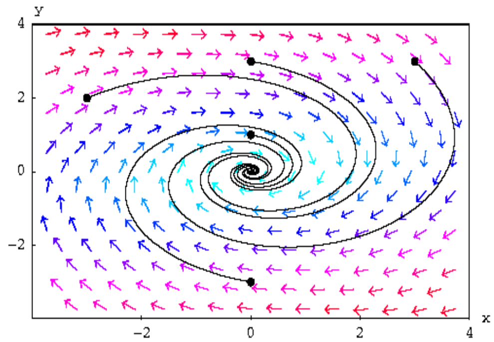
8. Find the eigenvalues and eigenvectors of the system $\vec{x}' = \begin{bmatrix} 4 & 1 \\ 6 & -1 \end{bmatrix} \vec{x}$. Draw the phase plane and plot several sample trajectories. Is the origin a repeller, attracter, or saddle point.

9. Verify that the equation $(1 + ye^{xy})dx + (2y + xe^{xy})dy = 0$ is exact. Then find the general solution.

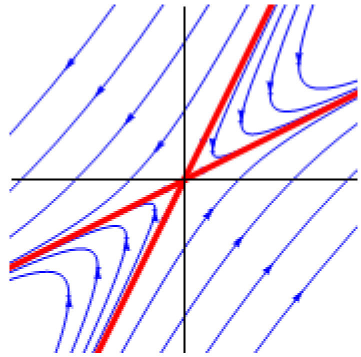
10. Use the method of integrating factors to find the particular solution for $xy' = 2y + x^3 \cos x$, $y(\pi) = 0$.

11. Rewrite the equation $y' + \frac{6}{x} y = 3y^{4/3}$ as a linear equation.

12. For each set of solution curves shown below, match the graphs with proposed solutions and characterize the system as containing a stable vector/orbit, origin attracts, origin repels, origin is a saddle point. (12 points)



a.



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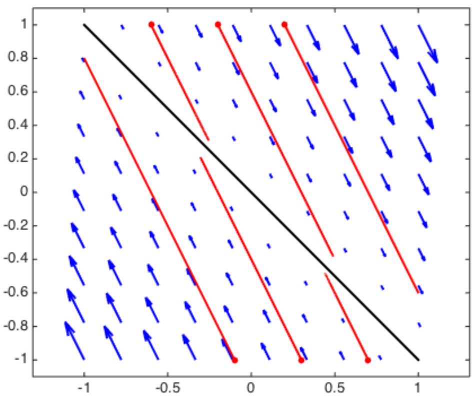
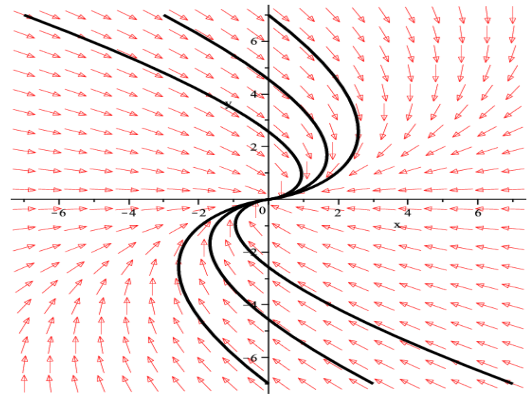


b.



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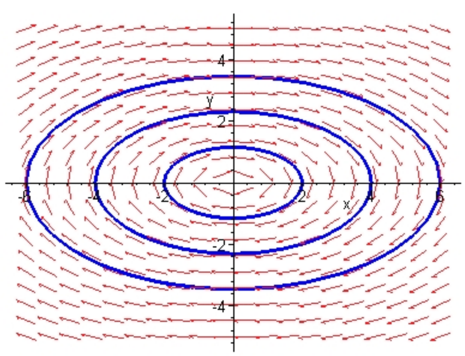
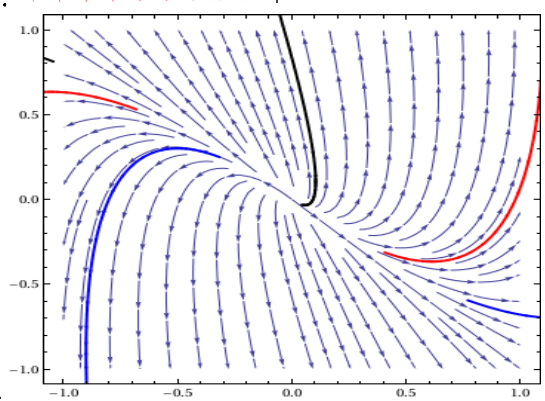


c.



f.

i. $\vec{x}(t) = c_1 \begin{pmatrix} 2\cos t \\ \sin t \end{pmatrix} + c_2 \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}$

ii. $\vec{x}(t) = c_1 e^{-t} \begin{pmatrix} 3\cos t + \sin t \\ 2\sin t \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} -3\sin t + cost \\ -2\cos t \end{pmatrix}$

iii. $\vec{x}(t) = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t}$

iv. $\vec{x}(t) = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 4 \end{pmatrix} e^{-2t}$

v. $\vec{x}(t) = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{2t}$

vi. $\vec{x}(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-3t}$

13. Solve $y' + 2xy^2 = 0$ by separation of variables.

14. Classify each differential equation as i) linear or nonlinear, ii) state its order.
   a. $yy' = x(y^2 + 1)$

   b. $\frac{d^4y}{dx^4} = y \cos x$

   c. $2\sqrt{x}\frac{du}{dx} + \left(\frac{du}{dx}\right)^2 = 2xu$

   d. $y^{(5)} + y'' = e^y \tan x$

15. Solve the second-order ODEs for the general solution.
   a. $y'' - 2y' + 2y = 0$

   b. $2y'' - y' - 2y = 0$

16. The table below gives the solution to the second order constant coefficient homogeneous equation, and the forcing function $F(x)\ or\ F(t)$. Determine the Ansatz for the method of undetermined coefficients in each case.

	y_1	y_2	$F(x)\ or\ F(t)$	Ansatz
a.	e^{-4x}	$e^{0.1x}$	$2\sinh 3x$	
b.	$e^x \cos x$	$e^x \sin x$	$e^x \sin x$	
c.	e^x	$e^{-x/3}$	$e^x + 7x^3$	

17. Use the table of Laplace transforms to find Laplace transforms or inverse Laplace transforms as indicated. (4 points each)
   a. $\mathcal{L}\{(1+2t)^2\}$

   b. $\mathcal{L}\{e^{-2t} \sin 3t\}$

   c. $\mathcal{L}\left\{\frac{1}{2}\int_0^t (t-\tau)^3 \sin 2\tau\, d\tau\right\}$

   d. $\mathcal{L}^{-1}\left\{\frac{1}{2} - \frac{2}{s^5}\right\}$

   e. $\mathcal{L}^{-1}\left\{\frac{9-17s}{s^2+81}\right\}$

   f. $\mathcal{L}^{-1}\left\{\frac{1}{(s-3)(s^2+4)(s^2-1)}\right\}$

   g. $\mathcal{L}^{-1}\left\{\frac{e^{-\pi}}{s^2+1}\right\}$

18. Use Laplace transforms to solve the IVP $y'' + 4y' - 12y = e^{-2t}, y(0) = 0,\ y'(0) = 1.$

19. We want to approximate the solution to $y' = x + \sqrt[3]{y}$ at the point $x = 3$ in 10 steps. Given that $y(0) = 1$, compute the first 3 steps of the approximation with Euler's method.

20. A 1000L tank initially contains only pure water. A hose begins adding to the tank at a rate of 5L/min with a concentration of iodine salt of 40g/L. The well-mixed solution flows out of the tank at a rate of 6L/min. Find an equation that models the amount of iodine in the tank after time t. Find the maximum amount of iodine in the tank (if one exists).

21. Prove that $\vec{x} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} e^{2t}$ satisfies the differential equation $\vec{x}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \vec{x}$.

22. Determine if the set of functions forms a fundamental set.
    a. $e^t \sin t, e^t \cos t$

    b. $\cosh t, \sinh t$

23. Use reduction of order to find the second solution to the equation $(x-1)y'' - xy' + y = 0$, $y_1 = e^x$.

24. Find the particular solution $y'' + 2y' + 5y = 3\sin 2t, y(0) = 1, y'(0) = 3$ using:
    a. The method of undetermined coefficients
    b. Variation of parameters

25. A spring with a 4-kg mass has natural length 1 m and is maintained stretched to a length of 1.3 m by a force of 24.3 N. If the spring is compressed to a length of 0.8 m and then released with zero velocity, set up the second order linear IVP needed to solve the system, then solve it. You may round solutions to 4 decimal places.

26. A metal pan is removed from an oven at a temperature of 425-degrees. After 2 minutes, the pan temperature has fallen to 350-degrees.
    a. If the room temperature is 77-degrees, write a differential equation that models the situation, and then solve for the equation at time *t*.
    b. How long will it take for the temperature to fall to 120-degrees?