MTH 291 Practice Exam # 1 Key

1. $y'(x) = \frac{1}{x+c}$ $\quad e^y = e^{\ln(x+c)} = x+c$

$$e^y y' = (x+c)\cdot \frac{1}{x+c} = 1 \checkmark$$

$y(0) = \ln(0+c)$

this will equal 0 when $c=1$

2. See attached graph, answers will vary

3. $y^2(xy'+y)\sqrt{1+x^4} = x$

$$xy'+y = \frac{x}{y^2\sqrt{1+x^4}} \Rightarrow xy' = \frac{x}{y^2\sqrt{1+x^4}} - y$$

$$\Rightarrow y' = \frac{x}{xy^2\sqrt{1+x^4}} - \frac{y}{x}$$

$$\Rightarrow y' = \frac{x - y^3\sqrt{1+x^4}}{xy^2\sqrt{1+x^4}}$$

conditions:

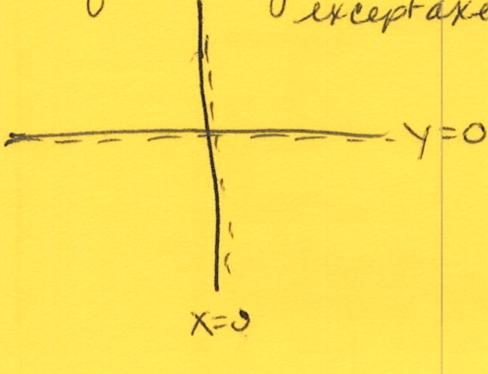
$f(x,y) = \frac{x - y^3\sqrt{1+x^4}}{xy^2\sqrt{1+x^4}}$ defined

$x \neq 0, y \neq 0 \quad 1+x^4 \geq 0$ (always true)

$f_y(x,y)$ defined $\quad f_y = \frac{(3y^2\sqrt{1+x^4})(xy^2\sqrt{1+x^4}) - (2xy\sqrt{1+x^4})(x-y^3\sqrt{1+x^4})}{x^2y^4(1+x^4)}$

defined everywhere except axes.

$$= \frac{3xy^4(1+x^4) - 2x^2y\sqrt{1+x^4} + 2xy^4(1+x^4)}{x^2y^4(1+x^4)}$$



$$= \frac{5xy^4(1+x^4) - 2x^2y\sqrt{1+x^4}}{x^2y^4(1+x^4)}$$

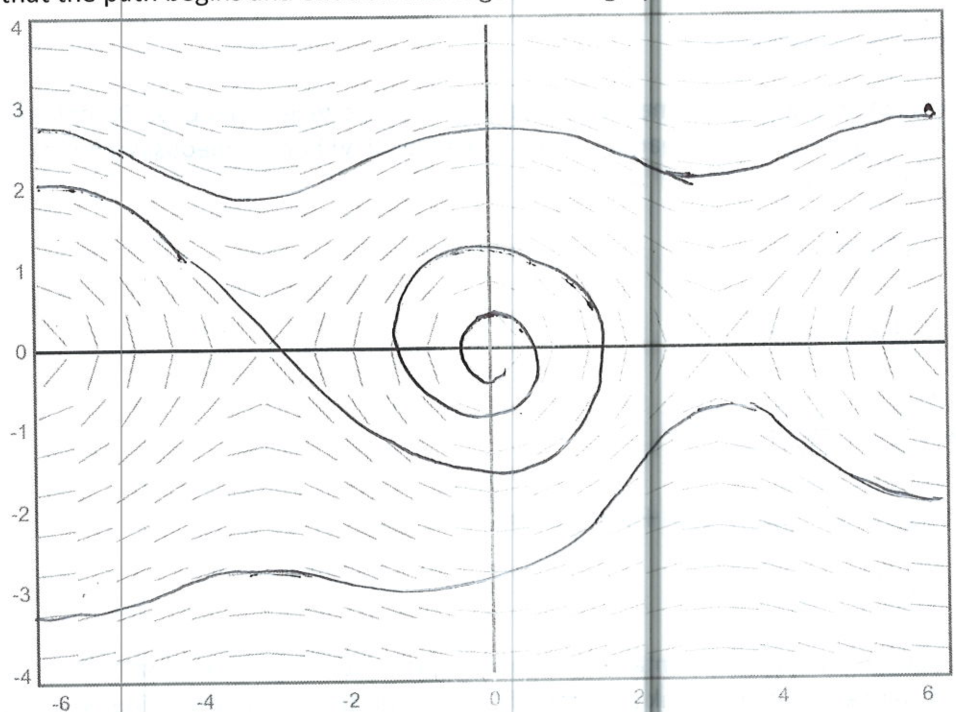
$x \neq 0, y \neq 0 \quad 1+x^4 \geq 0$

MTH 291, Practice Exam #1, Fall 2018 Name ______________________________

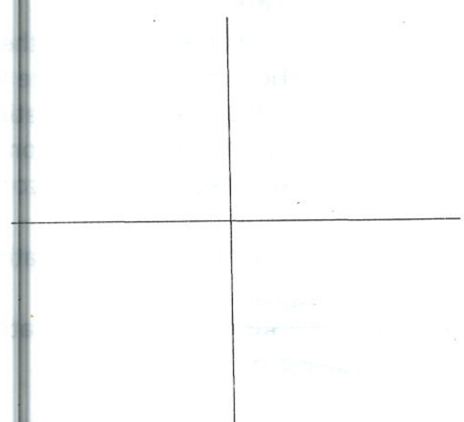
Instructions: Show all work. Answers without work required to obtain the solution will not receive full credit. Some questions may contain multiple parts: be sure to answer all of them. Give exact answers unless specifically asked to estimate.

1. Verify that $y(x) = \ln(x + C)$ is a solution to the differential equation $e^y y' = 1, y(0) = 0$.

2. Shown below is the slope field for an undamped pendulum. Plot at least three sample trajectories (integral solutions) with different behaviors. Track the path forward and backward in time (so that the path begins and ends on the edges of the graph.



3. Use the Existence and Uniqueness Theorem to determine the regions where a solution to the ODE $y^2(xy' + y)\sqrt{1 + x^4} = x$ is guaranteed to exist. Sketch the region in the plane. Be sure to check all conditions and show your work.



4. Solve $x^2 y' = 1 - x^2 + y^2 - x^2 y^2$ by separation of variables. [Hint: Factor by grouping.]

5. Classify each differential equation as i) linear or nonlinear, ii) state its order.
   a. $x^2 y'' + 5xy' + 4y = 0$

4. $x^2y' = 1 - x^2 + y^2 - x^2y^2$
$= (1-x^2) + y^2(1-x^2)$
$= (1+y^2)(1-x^2)$

$$\frac{\cancel{x^2}y'}{(1+y^2)\cancel{x^2}} = \frac{\cancel{(1+y^2)}}{\cancel{(1+y^2)}}\frac{(1-x^2)}{x^2} \Rightarrow \int \frac{y'}{1+y^2} = \int \frac{1}{x^2} - 1$$

$$\arctan y = -\frac{1}{x} - x + C$$
$$y = \tan\left(-\frac{1}{x} - x + C\right)$$

5. a. linear, 2nd order
b. linear, 2nd order
c. nonlinear, 3rd order
d. nonlinear, 5th order

6a. homogeneous. all terms contain y, y' or y''
b. nonhomogeneous term $5x^4$ does not contain y, or y'

7a. Bernoulli
b. separable
c. linear
d. separable
e. exact
f. homogeneous.

8. $\frac{dA}{dt} = \frac{1\text{ lbs}}{1\text{ gal}} \cdot \frac{5\text{ gal}}{1\text{ sec}} - \frac{A\text{ lbs}}{100+2t\text{ gal}} \cdot \frac{3\text{ gal}}{1\text{ sec}}$ $A(0) = 50$

$\frac{dA}{dt} = 5 - \frac{3A}{100+2t} \Rightarrow \frac{dA}{dt} + \frac{3}{100+2t}A = 5$

$\mu = e^{3\int \frac{1}{100+2t}dt}$
$= e^{\frac{3}{2}\ln(100+2t)}$
$= (100+2t)^{3/2}$

$(100+2t)^{3/2}\frac{dA}{dt} + 3(100+2t)^{1/2}A = 5(100+2t)^{3/2}$

8 cont'd.

$$\int\left[(100+2t)^{3/2}A\right]' = \int 5(100+2t)^{3/2}$$

$$(100+2t)^{3/2}A = \frac{\cancel{5}\cdot\frac{\cancel{2}}{\cancel{5}}}{\cancel{2}}(100+2t)^{5/2} + C$$

$$A = \frac{(100+2t)^{5/2}+C}{(100+2t)^{3/2}} = 100+2t+\frac{C}{(100+2t)^{3/2}}$$

$$50 = 100+2\cancel{(0)} + \frac{C}{(100+2\cancel{(0)})^{3/2}}$$

$$-50 = \frac{C}{(100)^{3/2}} \Rightarrow C = -50(1000) = -50{,}000$$

a. $A = 100+2t - \dfrac{50{,}000}{(100+2t)^{3/2}}$

$100+2t = 400 \Rightarrow 2t = 300 \Rightarrow t = 150$

can fill for 150 seconds before overflowing

b. $A(150) = 100+2(150) - \dfrac{50{,}000}{(100+2(150))^{3/2}} = 393.75$ lbs of salt

c. concentration $= \dfrac{\text{total salt}}{\text{amount of water}}$

$$C(t) = \frac{A(t)}{100+2t}$$

d. when tank is full $C(t) = .984375$ lbs/gal

e. 90% of $.984375 = .8859375$

$$C(t) = \frac{100+2t-50{,}000/(100+2t)^{3/2}}{100+2t} = 1-\frac{50{,}000}{(100+2t)^{5/2}}$$

$$1-\frac{50{,}000}{(100+2t)^{5/2}} = .8859375 \qquad t = 40.3 \text{ seconds}$$

9. $xy' - 3y = 2x^4e^x$

$y' - \frac{3}{x}y = 2x^3e^x \qquad \mu = e^{\int -\frac{3}{x}dx} = e^{-3\ln x} = x^{-3}$

$x^{-3}y' - 3x^{-4}y = 2 e^x$

$\int (x^{-3}y)' = \int 2 e^x$

$x^{-3}y = 2e^x + C$

$y = 2x^3e^x + Cx^3$

10. $x^2y' + 2xy = 5y^4 \;\Big]\, y^{-4}(-3) \qquad z = y^{-3} \Rightarrow z' = -3y^{-4}y'$

$x^2y^{-4}y' + 2xy^{-3} = 5 \;\Big]\,(-3)$

$-3x^2y^{-4}y' - 6xy^{-3} = -15$

$x^2z' - 6xz = -15$

$z' - \frac{6}{x}z = -\frac{15}{x^2} \;\Big]\, x^{-6} \qquad \mu = e^{\int -\frac{6}{x}dx} = e^{-6\ln x} = x^{-6}$

$x^{-6}z' - 6x^{-7}z = -15x^{-8}$

$\int (x^{-6}z)' = \int -15x^{-8}$

$x^{-6}z = \frac{15}{7}x^{-7} + C$

$z = \frac{15}{7x} + Cx^6$

$y^{-3} = \frac{15}{7x} + Cx^6$

$\frac{1}{y^3} = \frac{15}{7x} + Cx^6$

11. $\frac{\partial M}{\partial y} = 2y \qquad \frac{\partial N}{\partial x} = 2y$ ✓

$\int 2x + y^2\,dx = \frac{2x^2}{2} + xy^2 + f(y)$

$\int 2xy\,dy = xy^2 + g(x)$

$\varphi(x,y) = x^2 + xy^2 + K$

12. $y' = \dfrac{y}{x+2y}$ Order 1

$y = vx \rightarrow v = \frac{y}{x}$

$y' = v'x + v$

$v'x + v = \dfrac{vx}{x+2vx}$

$v'x + v = \dfrac{\cancel{x}(v)}{\cancel{x}(1+2v)}$

$v'x + \cancel{v} = \dfrac{v}{1+2v} - v$

$v'x = \dfrac{v - v(1+2v)}{1+2v} = \dfrac{\cancel{v} - \cancel{v} - 2v^2}{1+2v}$

$v'x = \dfrac{-2v^2}{1+2v}$

$dv \cdot \dfrac{1+2v}{-\cancel{2}v^2} = -2x\,dx$

$\int \frac{1}{v^2} + \frac{2}{v}\,dv = \int -2x\,dx$

$-\frac{1}{v} + 2\ln v = -x^2 + C$

$-\frac{x}{y} + 2\ln(y/x) = -x^2 + C$

13. $\dfrac{dy}{dt} = (y-2)(y-1)^2(y+1)$

14. stable and $y = M$, w/ $M > 0$

15. $y' = \dfrac{y^2}{x}$ $\qquad \frac{2-1}{5} = 0.2$

$n=0 \quad x_0 = 1,\ y_0 = 1 \quad y' = m = \frac{1^2}{1} = 1 \quad y_1 = 1 + 1(0.2) = 1.2$

$n=1 \quad x_1 = 1.2,\ y_1 = 1.2 \quad y' = \frac{1.2^2}{1.2} = 1.2 \quad y_2 = 1.2 + 1.2(0.2) = 1.44$

$n=2 \quad x_2 = 1.4,\ y_2 = 1.44 \quad y' = \frac{1.44^2}{1.4} = 1.48 \quad y_3 = 1.44 + 1.48(.2) = 1.736$

$n=3 \quad x_3 = 1.6,\ y_3 = 1.736 \quad y' = \frac{1.736^2}{1.6} = 1.884 \quad y_4 = 1.736 + 1.884(.2) = 2.113$

$n=4 \quad x_4 = 1.8,\ y_4 = 2.113 \quad y' = \frac{2.113^2}{1.8} = 2.48 \quad y_5 = 2.113 + 2.48(.2) = 2.609$

$n=5 \quad x_5 = 2 \quad \boxed{y_5 = 2.609}$

$y(2) \approx 2.609$

16. $\frac{dy}{dx} = x\cos x$ $\quad y(0) = 1 \quad \Delta t = h = 0.1$

$n=0 \quad x_0 = 0,\ y_0 = 1$

$k_{01} = 0.1 \cdot 1.0 \cdot \cos 0 = 0.1$

$k_{02} = 0.1\,(1.05)\cos(0.05) = 0.105$

$k_{03} = 0.1(1.052)\cos(.05) = 0.105$

$k_{04} = .1\,(1.105)\cos(.1) = 0.1099$

$y_1 = 1 + \frac{1}{6}(.1 + 2(0.105) + 2(0.105) + .1099)$

$y_1 = 1 + \frac{1}{6}(.6299) =$

1.105

$n=1 \quad x_1 = 0.1 \quad y_1 = 1.105$

$k_{11} = 0.1((1.105)\cos(0.1) = 0.1099$

$k_{12} = 0.1(1.055)\cos(0.15) = 0.1043$

$k_{13} = 0.1\,(1.052)\cos(0.15) = 0.1040$

$k_{14} = 0.1\,(1.104)\cos(.2) = 0.1082$

$y_2 = 1.105 + \frac{1}{6}(.1099 + 2(0.1043) + 2(0.1040) + 0.1082)$

$\Rightarrow y_2 = 1.105 + \frac{1}{6}(1.2345)$

$= 1.31075$

$y(.2) \approx 1.31$

know this formula.

R-K $\rightarrow$

$k_1 = hf(x_n, y_n)$

$k_2 = hf(x_n + \frac{1}{2}h,\ y_n + \frac{1}{2}k_1)$

$k_3 = hf(x_n + \frac{1}{2}h,\ y_n + \frac{1}{2}k_2)$

$k_4 = hf(x_n + h,\ y_n + k_3)$

17. $\frac{\partial M}{\partial y} = 0 \qquad \frac{\partial N}{\partial x} = e^x \cot y + 0$

$e^x dx + (e^x \cot y + 2y \csc y)\,dy = 0 \;\Big]\times \sin y$

$e^x \sin y\,dx + (e^x \cos y + 2y)\,dy$

$\frac{\partial M}{\partial y} = e^x \cos y \qquad \frac{\partial N}{\partial x} = e^x \cos y$ ✓

$\int e^x \sin y\,dx = e^x \sin y + f(y)$

$\int e^x \cos y + 2y\,dy = e^x \sin y + y^2 + g(x)$

$\varphi(x,y) = e^x \sin y + y^2 + K$

18. $\frac{\partial M}{\partial y} = \;\; - kx \qquad k=2$

$\frac{\partial N}{\partial x} = -2x$

19. $k(T-72) = \frac{dT}{dt}$

$\int k\,dt = \int \frac{dT}{T-72}$

$kt + C = \ln|T-72|$

$A_0 e^{kt} = T - 72$

$T(0) = 200$

$T = 72 + A_0 e^{kt}$

$A_0 = 128$

$T(1) = 180$

$180 = 72 + 128e^{k(1)}$

$\frac{108}{128} = e^k \Rightarrow k = -.1699$

$T(t) = 72 + 128e^{-.1699t}$

$120 = 72 + 128e^{-.1699t}$

$t = 5.77$ minutes

20. half-life = 11.2 hours.

$\frac{dA}{dt} = kA$

$A(t) = A_0 e^{kt}$ peak = 100%

$50\% = 100\% e^{k(11.2)}$ $k = -.061888$

$5\% = 100\% e^{-.061888t} \rightarrow t = 48.4$ hours