MTH 174 Practice Exam #3 Key

1. $\lim_{n\to\infty} \cos\frac{2}{n} = \cos\left[\lim_{n\to\infty} \frac{2}{n}\right] = \cos(0) = 1$

2. $\frac{(2n+2)!}{(2n)!} = \frac{(2n+2)(2n+1)\cancel{(2n)!}}{\cancel{(2n)!}} = (2n+2)(2n+1)$

3. bounded between 0 and 4

$\frac{3n}{n+2} < 4 \rightarrow 3n < 4n+8 \rightarrow n > -8$
Since $n > 0$, always true

$\frac{3n}{n+2} > 0$ $\quad \overset{+}{} \underset{-2}{|} \overset{-}{} \underset{0}{|} \overset{+}{}$ for $n>0$, $\frac{3n}{n+2} > 0$

$f(x) = \frac{3x}{x+2} \quad f'(x) = \frac{3(x+2) - 1(3x)}{(x+2)^2} = \frac{\cancel{3x}+6-\cancel{3x}}{(x+2)^2} = \frac{6}{(x+2)^2}$

this is always positive, so always increasing

thus a_n is both bounded and monotonic

4. a. $\sum_{n=0}^{\infty} 5\left(\frac{2}{3}\right)^n \quad a=5,\ r=\frac{2}{3} \quad = \frac{5}{1-\frac{2}{3}} = \frac{5}{1/3} = 15$

b. $\sum_{n=1}^{\infty} \frac{4}{n(n+2)} \quad \frac{A}{n} + \frac{B}{n+2} =$ $\quad An + 2A + Bn = 4$

$A + B = 0$
$2A = 4$
$A = 2,\ B = -2$

$\sum_{n=1}^{\infty} \left(\frac{2}{n} - \frac{2}{n+2}\right)$

Sum $= \frac{2}{1} + \frac{2}{2} - \lim_{n\to\infty}\left(\cancel{\frac{2}{n+1}}^{0} + \cancel{\frac{2}{n+2}}^{0}\right) = 2+1 = 3$

5. $S_n = \frac{1}{1} + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}$

$\lim_{n\to\infty} \frac{1}{1} + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2} = 1 + \frac{1}{2} + 0 - 0 = \frac{3}{2}$

$\lim_{n\to\infty} \frac{\ln n}{n} =$
$\lim_{n\to\infty} \frac{\frac{1}{n}}{1} = 0$

6. $\int_1^{\infty} \frac{\ln n}{n^2}\,dn \quad u = \ln n \quad dv = \frac{1}{n^2}\,dn$
$du = \frac{1}{n} \quad v = -\frac{1}{n}$

$-\frac{\ln n}{n}\Big|_1^{\infty} + \int_1^{\infty} \frac{1}{n^2}\,dn = -\frac{\ln n}{n} - \frac{1}{n}\Big|_1^{\infty}$

$\lim_{n\to\infty} \cancel{\frac{-\ln n}{n}}^{0} - \cancel{\frac{1}{n}}^{0} + \cancel{\frac{\ln 1}{1}}^{0} + \frac{1}{1} = 1$ Converges

7a. $\frac{1}{\sqrt{n^3+1}} \leq \frac{1}{\sqrt{n^3}} = \frac{1}{n^{3/2}}$ since $\sqrt{n^3+1} > \sqrt{n^3}$

we know from p-series test that $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges so $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}}$

converges by direct comparison

b. $\sum_{n=1}^{\infty} \frac{n}{(n+1)2^{n-1}}$ $\quad \lim_{n\to\infty} \left| \frac{\frac{n+1}{(n+2)2^n}}{\frac{n}{(n+1)2^{n-1}}} \right| = \lim_{n\to\infty} \left| \frac{n+1}{(n+2)2^n} \cdot \frac{(n+1)2^{n-1}}{n} \right| =$

$\lim_{n\to\infty} \left| \frac{\cancel{(n+1)(n+1)}}{\cancel{n(n+2)}} \cdot \frac{\cancel{2^{n-1}}}{\cancel{2^n}} \right| = \frac{1}{2} < 1$ So by the ratio test, the sum converges

c. $\sum_{n=2}^{\infty} \frac{1}{5^n+1}$ $\quad b_n = \frac{1}{5^n}$ (sum converges by geometric series test)

$\lim_{n\to\infty} \frac{\frac{1}{5^n+1}}{\frac{1}{5^n}} = \lim_{n\to\infty} \frac{5^n}{5^n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{5^n}} = 1$

thus $\sum a_n$ and $\sum b_n$ converge together.

d. $\sum_{n=0}^{\infty} \frac{6^n}{(n+1)^n}$ $\quad \lim_{n\to\infty} \sqrt[n]{\frac{6^n}{(n+1)^n}} = \lim_{n\to\infty} \frac{6}{n+1} = 0 < 1$ So, this

series converges by root test

e. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sqrt{n}}{n+2}$ $\quad \lim_{n\to\infty} \frac{\sqrt{n}}{n+2} = 0$ Thus sum converges by alternating series test

f. $\sum_{n=1}^{\infty} \frac{ne^{\sqrt{n}}}{n^2+1{,}000{,}000}$ $\quad \lim_{n\to\infty} \frac{ne^{\sqrt{n}}}{n^2+10^6} = \infty$ ($e^{\sqrt{n}}$ blows up fastest)

Therefore, sum diverges by nth term test

8.a. $\sum_{n=0}^{\infty} \frac{2^n}{n!}$ $\quad \lim_{n\to\infty} \left| \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} \right| = \lim_{n\to\infty} \left| \frac{2}{n+1} \right| = 0 < 1$

ratio test $\quad$ Series converges by the ratio test. $\quad$ factorials work best w/ ratio

8b. $\sum_{n=2}^{\infty} \frac{n}{(\ln n)^n}$ $\lim_{n\to\infty} \sqrt[n]{\frac{n}{(\ln n)^n}} = \lim_{n\to\infty} \frac{\sqrt[n]{n}}{\ln n} = \frac{\lim_{n\to\infty} \sqrt[n]{n}}{\lim_{n\to\infty} \ln n} = \frac{1}{\underbrace{\lim_{n\to\infty} \ln n}_{=\infty}} = 0$

root test

$0 < 1$ so the series converges by root test

powers of n, n^n suggest this is a good test

c. $\sum_{n=1}^{\infty} \left(\frac{1}{n^2} - \frac{n}{3^n+1}\right) = \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^2}} - \underbrace{\sum_{n=1}^{\infty} \frac{n}{3^n+1}}$

converges by p-series $p=2 \Rightarrow p>1$

compare w/ $\frac{n}{3^n}$ (converges by ratio) using limit comparison

$\sum_{n=1}^{\infty} \frac{n}{3^n}$ ← b_n

$\lim_{n\to\infty} \left| \frac{n+1}{3^{n+1}} \cdot \frac{3^n}{n} \right| = \frac{1}{3} < 1$ Converges by ratio test

$\lim_{n\to\infty} \frac{\frac{n}{3^n+1}}{\frac{n}{3^n}} = \lim_{n\to\infty} \frac{\cancel{n}}{3^n+1} \cdot \frac{3^n}{\cancel{n}} = \lim_{n\to\infty} \frac{3^n}{3^n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{3^n}} = 1$

Converges since b_n converges

Since both component converge separately, they converge together

9. $\sum_{n=0}^{\infty} \frac{\cos n\pi}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$

$\lim_{n\to\infty} \frac{1}{n+1} = 0$ converges by alternating series test

but $\sum_{n=0}^{\infty} \frac{1}{n+1}$ diverges by comparison w/ harmonic series $\frac{1}{n}$

$\lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n+1} = 1$ Since $\sum \frac{1}{n}$ diverges, so does $\sum \frac{1}{n+1}$

therefore, the series $\sum_{n=0}^{\infty} \frac{\cos n\pi}{n+1}$ converges conditionally

10. $\sum_{n=1}^{\infty} \frac{4}{n(n+2)}$

a. $\lim_{n\to\infty} \left| \frac{\frac{4}{(n+1)(n+3)}}{\frac{4}{n(n+2)}} \right| = \lim_{n\to\infty} \left| \frac{\cancel{4}}{(n+1)(n+3)} \cdot \frac{n(n+2)}{\cancel{4}} \right| = \lim_{n\to\infty} \left| \frac{n(n+2)}{(n+1)(n+3)} \right| = 1$

Since this limit is 1, it is inconclusive

b. decreasing means $f'(x) < 0$

$f(x) = \frac{4}{n^2+2n}$ $\quad f'(x) = \frac{0(n^2+2n) - (2n+2)(4)}{n^2(n+2)^2} = \frac{-8(n+1)}{n^2(n+2)^2} \leq 0$

for all $n \geq 1$

c. $\int_1^{\infty} \frac{4}{n(n+2)}\, dn = \int_1^{\infty} \frac{2}{n} - \frac{2}{n+2}\, dn = 2\ln n - 2\ln(n+2) \Big|_1^{\infty}$

$\lim_{n\to\infty} 2\ln\left|\frac{n}{n+2}\right| - 2\ln\left|\frac{1}{3}\right| = 2(0) - 2\ln\left|\frac{1}{3}\right|$

$\lim_{n\to\infty} \frac{n}{n+2} = 1$ and $\ln 1 = 0$

Since this is finite, the series converges

11. a. $\lim_{n\to\infty} \left| \frac{\cancel{(-1)^{n+1}}(x+1)^{\cancel{n+1}}}{n+1} \cdot \frac{n}{\cancel{(-1)^n}\cancel{(x+1)^n}} \right| = \underbrace{\lim_{n\to\infty} \left|\frac{n}{n+1}\right|}_{=1} \cdot \lim_{n\to\infty} |x+1| = |x+1| < 1$

$$\begin{array}{ccccc} -1 & < & x+1 & < & 1 \\ -1 & & -1 & & -1 \\ \hline -2 & < & x & < & 0 \end{array}$$

$(-2, 0]$

$\sum_{n=1}^{\infty} \frac{(-1)^n(-1)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$ harmonic series diverges

$\sum_{n=1}^{\infty} \frac{(-1)^n (1)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges by alternating series test

$\boxed{(-2, 0]}$

b. $\lim_{n\to\infty} \left| \frac{(n+1)!}{n!} \left(\frac{x}{2}\right)^{\cancel{n+1}} \cancel{\left(\frac{2}{x}\right)^n} \right|$

$= \lim_{n\to\infty} \left|(n+1)\left(\frac{x}{2}\right)\right| = \infty > 0$ for all $x \neq 0$

$\{0\}$

(5)

12. a. $f(x) = \dfrac{4}{2+3x} = \dfrac{4}{2+3(x-3)+9} = \dfrac{4}{11+3(x-3)} = \dfrac{4/11}{1+\frac{3}{11}(x-3)}$

$a = \frac{4}{11}, \quad r = \frac{3}{11}(x-3)$

$f(x) = \sum_{n=0}^{\infty} \frac{4}{11}\left[\frac{3}{11}(x-3)\right]^n = \sum_{n=0}^{\infty} \frac{4 \cdot 3^n}{11^{n+1}}(x-3)^n$

b. $\sum_{n=0}^{\infty} a x^n = \dfrac{a}{1-x} \Rightarrow \sum_{n=1}^{\infty} a n x^{n-1} = \sum_{n=0}^{\infty} a(n+1)x^n = \dfrac{a}{(1-x)^2}$

$a = 4x, \quad r = -x$

$f(x) = \dfrac{4x}{(1+x)^2} = \sum_{n=0}^{\infty} 4x(n+1)(-x)^n = \sum_{n=0}^{\infty} 4(-1)^n(n+1)x^{n+1}$

13. a. @1

$f(x) = \ln x \; (0), \quad f'(x) = \frac{1}{x} \; (1), \quad f''(x) = \frac{-1}{x^2} \; (-1), \quad f'''(x) = \frac{2}{x^3} \; (2), \quad f^{iv}(x) = \frac{-6}{x^4} \; (-6)$

$f^{v}(x) = \frac{24}{x^5} \; (24), \quad f^{vi}(x) = \frac{-120}{x^6} \; (-120)$

$P_6 = 0 + 1(x-1) - \frac{1}{2}(x-1)^2 + \frac{2}{6}(x-1)^3 - \frac{6}{24}(x-1)^4 + \frac{24}{120}(x-1)^5 - \frac{120}{720}(x-1)^6$

b. $f'(x) = 3e^{3x}, \quad f''(x) = 9e^{3x}, \quad f'''(x) = 27e^{3x}, \quad f^{iv}(x) = 81e^{3x}$

$f(x) = e^{3x}$ @0 $\quad 1, 3, 9, 27, 81$

$P_4 = 1 + 3x + \frac{9}{2}(x)^2 + \frac{27}{6}x^3 + \frac{81}{24}x^4$

14. $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x \qquad e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$

$1 - x + \frac{x^2}{2} - \frac{x^3}{6} = P_3 \qquad P_3(.1) \approx .9048$