MATH 240, Exam #4 Practice, Summer 2017 Name KEY

Instructions: Show all work. Answers without work required to obtain the solution will not receive full credit. Some questions may contain multiple parts: be sure to answer all of them. Give exact answers unless specifically asked to estimate.

1. Determine whether the vector field $\vec{F}(x,y,z) = \frac{y}{x^2}\hat{\imath} - \frac{1}{x}\hat{\jmath} + 2\hat{k}$ is conservative. If it is, find a potential function for the vector field. (10 points)

$\int \frac{y}{x^2}dx = -\frac{y}{x} + C_1(y,z)$

$\int -\frac{1}{x}dy = -\frac{y}{x} + C_2(x,z)$

$\int 2\,dz = 2z + C_3(x,y)$

yes it is conservative and a potential function exists

$\varphi(x,y,z) = -\frac{y}{x} + 2z + K$

$$\nabla \times F = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y/x^2 & -1/x & 2 \end{vmatrix} = (0-0)\hat{\imath} - (0-0)\hat{\jmath} - \left(-\frac{1}{x^2} + \frac{1}{x^2}\right)\hat{k} = \vec{0}$$

2. Evaluate the line integral $\int_C (x^2+y^2+z^2)ds$ along the path $C: \vec{r}(t) = \cos(2t)\,\hat{\imath} + 3t\hat{\jmath} + \sin(2t)\,\hat{k}$. (10 points) $[0, 2\pi]$

$r'(t) = -2\sin 2t\,\hat{\imath} + 3\hat{\jmath} + 2\cos 2t\,\hat{k}$

$\|r'(t)\| = \sqrt{4\sin^2 2t + 9 + 4\cos^2 2t} = \sqrt{4+9} = \sqrt{13}\,dt$

$\int_0^{2\pi} (\cos^2 2t + 9t^2 + \sin^2 2t)\sqrt{13}\,dt =$

$\sqrt{13}\int_0^{2\pi} 9t^2 + 1\,dt = \sqrt{13}\left[3t^3 + t\right|_0^{2\pi} =$

$\sqrt{13}\left[3(2\pi)^3 + 2\pi\right] = \boxed{\sqrt{13}\left[24\pi^3 + 2\pi\right]}$

3. Find the work done by the force field $\vec{F}(x,y,z) = x\hat{\imath} + y\hat{\jmath} - \frac{1}{2}\hat{k}$ on a particle as it moves along the helix given by $\vec{r}(t) = 4\cos t\,\hat{\imath} + 3t\hat{\jmath} + 4\sin t\,\hat{k}$ from the point $(4,0,0)$ to $(-4, 9\pi, 0)$. (15 points)

$$\int x\,dx = \frac{1}{2}x^2 + C_1(y,z)$$

$$\int y\,dy = \frac{1}{2}y^2 + C_2(x,z)$$

field is conservative

$$\int -\frac{1}{2}\,dz = -\frac{1}{2}z + C_3(x,y)$$

$$\varphi(x,y,z) = \frac{1}{2}x^2 + \frac{1}{2}y^2 - \frac{1}{2}z + K$$

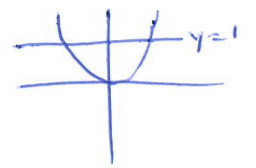
$$\int_C \vec{F}\cdot d\vec{r} = \varphi(-4, 9\pi, 0) - \varphi(4,0,0) =$$

$$\frac{1}{2}(-4)^2 + \frac{1}{2}(9\pi)^2 - \frac{1}{2}(0) - \left(\frac{1}{2}(4)^2 - 0^2 - 0^2\right)$$

$$\cancel{8} + \frac{81}{2}\pi^2 - \cancel{8} = \boxed{\frac{81}{2}\pi^2}$$

4. Use Green's Theorem $\oint_C (Mdx + Ndy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$ to evaluate $\oint_C [(x^2 - y^2)dx + xydy]$ where C is the boundary of the region R bounded by $y = x^2$, $y = 1$. (16 points)

$$\frac{\partial M}{\partial y} = -2y \qquad \frac{\partial N}{\partial x} = y$$



$$\int_{-1}^{1}\int_{x^2}^{1} y + 2y\,dy\,dx = \int_{-1}^{1}\int_{x^2}^{1} 3y\,dy =$$

$$\int_{-1}^{1} \frac{3}{2}y^2\Big|_{x^2}^{1} dx = \int_{-1}^{1} \frac{3}{2} - \frac{3}{2}x^4\,dx = 2\int_0^1 \frac{3}{2} - \frac{3}{2}x^4\,dx$$

even

$$= 2\left[\frac{3}{2}x - \frac{3}{2}\cdot\frac{1}{5}x^5\Big|_0^1 = 2\left[\frac{3}{2} - \frac{3}{10}\right] = \boxed{\frac{12}{5}}\right.$$

5. a. Find a parametrization of the cylinder $x^2 + y^2 = 1$ on the interval $0 \le z \le 1$. (10 points)

$x = \cos u \quad y = \sin u \quad z = v$

$\vec{r}(u,v) = \cos u\,\hat{\imath} + \sin u\,\hat{\jmath} + v\,\hat{k} \qquad u \in [0, 2\pi],\ v \in [0,1]$

b. Use that parametrization to find the value of $\iint_R x^2 dS$. (10 points)

$\vec{r}_u = -\sin u\,\hat{\imath} + \cos u\,\hat{\jmath} + 0\hat{k}$

$\vec{r}_v = 0\hat{\imath} + 0\hat{\jmath} + 1\hat{k}$

$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -\sin u & \cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} = (\cos u)\hat{\imath} + \sin u\,\hat{\jmath} + 0\hat{k}$

$\|\vec{r}_u \times \vec{r}_v\| = \sqrt{\cos^2 u + \sin^2 u} = 1$

$\int_0^{2\pi} \int_0^1 \cos^2 u \; 1 \, dv\, du = \int_0^{2\pi} \frac{1}{2}(1 + \cos 2u) v \Big|_0^1 du =$

$\frac{1}{2} \int_0^{2\pi} 1 + \cos 2u \, du = \frac{1}{2}\left[u + \frac{1}{2}\sin 2u\right]_0^{2\pi} = \frac{1}{2}\left[2\pi + 0 - 0 - 0\right]$

$= \boxed{\pi}$

6. a. Find the divergence of $\vec{F} = x\hat{\imath} + 2y^2\hat{\jmath} + 3z^2\hat{k}$. (10 points)

$$\vec{\nabla} \cdot \vec{F} = 1 + 4y + 6z$$

b. Use the divergence Theorem $\iint_R \vec{F} \cdot \vec{n} dS = \iiint_V div\, \vec{F}\, dV$ to find the values of $\iint_R \vec{F} \cdot \vec{n} dS$ where S is the surface of the cylinder $x^2 + y^2 = 9, 0 \leq z \leq 1$ using the field above. (12 points)

$$x = 3\cos\theta$$
$$y = 3\sin\theta$$

$$\int_0^{2\pi}\int_0^3\int_0^1 (1 + 12\sin\theta + 6z)\, r\, dz\, dr\, d\theta =$$

$$\int_0^{2\pi}\int_0^3 rz + 12rz\sin\theta + 3z^2 r\Big|_0^1 dr\, d\theta =$$

$$\int_0^{2\pi}\int_0^3 r + 12r\sin\theta + 3r\, dr\, d\theta = \int_0^{2\pi}\int_0^3 4r + 12r\sin\theta\, dr\, d\theta$$

$$= \int_0^{2\pi} 2r^2 + 6r^2\sin\theta\Big|_0^3 d\theta = \int_0^{2\pi} 18 + 54\sin\theta\, d\theta$$

$$\left[18\theta + -54\cos\theta\right|_0^{2\pi} = 18(2\pi) - \cancel{54(1)} - 0 + \cancel{54(1)} =$$

$$\boxed{36\pi}$$

7. a. Find the curl of $\vec{F} = y^2\hat{\imath} + z^2\hat{\jmath} + x^2\hat{k}$. (9 points)

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2 & z^2 & x^2 \end{vmatrix} = (0-2z)\hat{\imath} - (2x-0)\hat{\jmath} + (0-2y)\hat{k}$$

$$= -2z\hat{\imath} - 2x\hat{\jmath} - 2y\hat{k}$$

b. Find the parametric equation of the surface defined by the triangle with vertices (1,0,0), (0,1,0), (0,0,1). (8 points)

$\langle -1, 1, 0\rangle, \ \langle 0, -1, 1\rangle$

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{vmatrix} = 1\hat{\imath} + 1\hat{\jmath} + 1\hat{k}$$

$\langle 1, 1, 1\rangle$

$$1(x-1) + 1(y-0) + 1(z-0) = 0$$

$$x + y + z = 1 \rightarrow z = 1 - x - y$$

$$\vec{r}(u,v) = u\hat{\imath} + v\hat{\jmath} + (1-u-v)\hat{k}$$

$u \in [0,1]$

$v \in [0,1]$

$\langle 1,1,1\rangle$

c. Use that to evaluate $\oint_C \vec{F} \cdot d\vec{r} = \iint_R (curl\ \vec{F}) \cdot \vec{n} dS$ for the surface defined above. (8 points)

$$\vec{\nabla} \times \vec{F} \cdot \vec{n} = -2z - 2x - 2y = -2(z+x+y) = -2(x+y+z)$$

$$= -2(u+v+(1-u-v)) = -2(\cancel{u}+\cancel{v}+1-\cancel{u}-\cancel{v}) = -2$$

$$\int_0^1\int_0^1 -2\, du\, dv = \boxed{-2}$$