MATH 240, Exam #3 Practice, Summer 2017 Name KEY

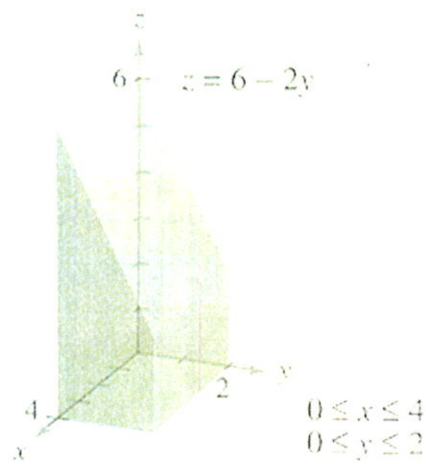
Instructions: Show all work. Answers without work required to obtain the solution will not receive full credit. Some questions may contain multiple parts: be sure to answer all of them. Give exact answers unless specifically asked to estimate.

1. Find the volume of the solid under the function $f(x, y) = 6 - 2y$ on the region bounded by $0 \le x \le 4, 0 \le y \le 2$ using a double integral. (9 points)

$$\int_0^4 \int_0^2 6 - 2y \, dy \, dx =$$

$$\int_0^4 6y - y^2 \Big|_0^2 dx = \int_0^4 12 - 4 \, dx =$$

$$\int_0^4 8 \, dx = 8x \Big|_0^4 = \boxed{32}$$

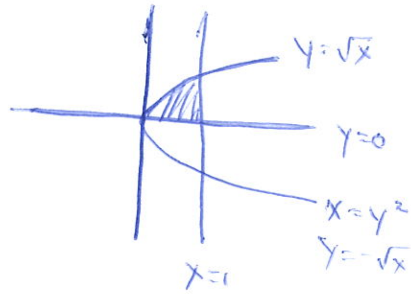


$y=1$ $\quad x=1$

2. The integral $\int_0^1 \int_{y^2}^1 \sqrt{x}\sin x \, dxdy$ can be evaluated only by changing the order of integration. Sketch the region of integration, reverse the order of integration, and evaluate the integral. (8 points)

$y=0$ $\quad x=y^2$

$$\int_0^1 \int_0^{\sqrt{x}} \sqrt{x}\,\sin x \, dy\, dx =$$



$$\int_0^1 (\sqrt{x}\sin x)\, y \Big|_0^{\sqrt{x}} dx =$$

$$\int_0^1 \sqrt{x}\sin x \cdot \sqrt{x}\, dx = \int_0^1 x\sin x\, dx$$

$u = x \quad dv = \sin x\,dx$

$du = dx \quad v = -\cos x$

$$-x\cos x\Big|_0^1 - \int_0^1 -\cos x\, dx =$$

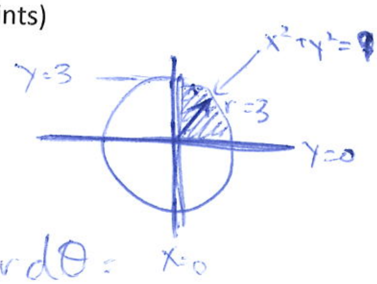
$$-x\cos x + \sin x\Big|_0^1 = -\cos 1 + \sin 1 - 0 =$$

$$\boxed{\sin 1 - \cos 1}$$

$y=3$ $\quad x=\sqrt{9-y^2}$

3. Evaluate $\int_0^3 \int_0^{\sqrt{9-y^2}} x dxdy$ by converting to polar coordinates. (8 points)

$y=0$ $\quad x=0$ $\quad x = r\cos\theta$



$$\int_0^{\pi/2}\int_0^3 r\cos\theta\, r\,dr\,d\theta = \int_0^{\pi/2}\int_0^3 r^2\cos\theta\, dr\,d\theta =$$

$$\int_0^{\pi/2} \frac{1}{3}r^3\cos\theta\Big|_0^3 d\theta = \int_0^{\pi/2} 9\cos\theta\, d\theta = 9\sin\theta\Big|_0^{\pi/2} = \boxed{9}$$

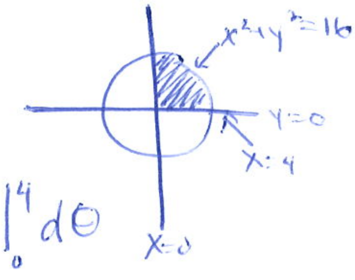
$y=\sqrt{16-x^2}$ $\quad z=\sqrt{x^2+y^2} \Rightarrow z=r$

4. Evaluate the integral $\int_0^4 \int_0^{\sqrt{16-x^2}} \int_0^{\sqrt{x^2+y^2}} \frac{1}{1+x^2+y^2} dzdydx$ in cylindrical coordinates. (10 points)

$y=0$

$x^2+y^2=r^2$ $\qquad \frac{1}{1+x^2+y^2}=\frac{1}{1+r^2}$

$x^2+y^2=16$



$$\int_0^{\pi/2}\int_0^4\int_0^r \frac{1}{1+r^2}\, r\, dz\, dr\, d\theta =$$

$$\int_0^{\pi/2}\int_0^4 \frac{rz}{1+r^2}\Big|_0^r dr\, d\theta = \int_0^{\pi/2}\int_0^4 \frac{r^2}{1+r^2} dr\, d\theta$$

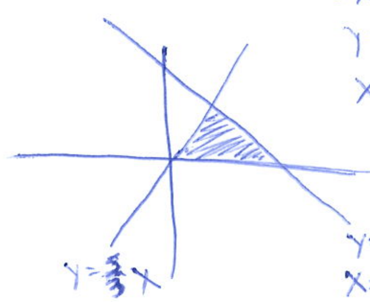
$$\int_0^{\pi/2}\int_0^4 1-\frac{1}{1+r^2}\, dr\, d\theta = \int_0^{\pi/2} r-\arctan r\Big|_0^4 d\theta$$

$$=\int_0^{\pi/2} 4-\arctan 4\, d\theta = (4-\arctan 4)\theta\Big|_0^{\pi/2} =$$

$$2\pi - \frac{\pi \arctan 4}{2}$$

5. Set up an iterated integral to find the area of the region bounded by $5x-3y=0, x+y=4$, $y=0$. Evaluate the integral to find the area. (10 points)

$5x=3y$
$y=\frac{5}{3}x$
$x+y=4$
$y=-x+4$



$$\int_0^{5/2}\int_{\frac{3}{5}y}^{-y+4} 1\, dx\, dy =$$

$$\int_0^{5/2} y\Big|_{3/5y}^{-y+4} dy =$$

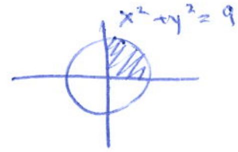
$-y+4=\frac{3}{5}y$
$-5y+20=3y$
$20=8y$
$y=\frac{5}{2}$

$$\int_0^{5/2} -y+4-\tfrac{3}{5}y\, dy = \int_0^{5/2} -\frac{8}{5}y+4\, dy$$

$$= -\frac{4}{5}y^2+4y\Big|_0^{5/2} = -\frac{4}{8}\left(\frac{5}{2}\right)^2+4(5/2)-0=$$

$$-5+10=\boxed{5}$$

6. Use a double integral to find the volume of the indicated solid: $z = x + y, x^2 + y^2 = 9$, first octant. (10 points)



$$\int_0^{\pi/2}\int_0^3 r(\cos\theta + \sin\theta)\, r\, dr\, d\theta =$$

$$\int_0^{\pi/2} \frac{1}{3}r^3(\cos\theta + \sin\theta)\Big|_0^3 d\theta =$$

$x + y = r\cos\theta + r\sin\theta$

$= r(\cos\theta + \sin\theta)$

$$9\int_0^{\pi/2} \cos\theta + \sin\theta \, d\theta = 9\left[\sin\theta - \cos\theta\right|_0^{\pi/2} =$$

$$9[1 - 0 - (0 - 1)] = \boxed{18}$$

7. Find the area of the surface given by $f(x, y) = 5 + 2x - 2y$ over the region R: rectangle with vertices $(0,0), (3,0), (3,1), (0,1)$. (10 points)

$0 \leq x \leq 3 \qquad 0 \leq y \leq 1$

$f_x = 2$

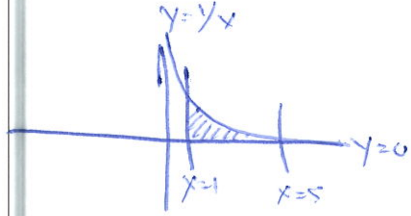
$f_y = -2$

$$\int_0^3\int_0^1 3\, dy\, dx = \qquad \sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1+4+4} = \sqrt{9} = 3$$

$$\int_0^3 3y\Big|_0^1 dx = \int_0^3 3\, dx = 3x\Big|_0^3 = \boxed{9}$$

8. Find the mass and centroid of the center of mass of the lamina bounded by the graphs of the equations $xy = 1, x = 1, x = 5, y = 0$ with the density $\rho = kxy$. (15 points)

$y = \frac{1}{x}$



$$M = \int_1^5 \int_0^{1/x} kxy\,dy\,dx = \int_1^5 \frac{k}{2} x y^2 \Big|_0^{1/x} dx$$

$$x \cdot \frac{1}{x^2}$$

$$= \int_1^5 \frac{k}{2} \cdot \frac{1}{x}\,dx = \frac{k}{2} \ln x \Big|_1^5 = \boxed{\frac{k}{2} \ln 5}$$

$$M_y = \int_1^1 \int_0^{1/x} kx^2 y\,dy\,dx = \int_1^5 \frac{k}{2} x^2 y^2 \Big|_0^{1/x} = \int_1^5 \frac{k}{2}\,dx = \frac{k}{2} x \Big|_1^5 =$$

$$x^2 \cdot \frac{1}{x^2}$$

$$\frac{k}{2}(5-1) = 2k$$

$$M_x = \int_1^5 \int_0^{1/x} kxy^2\,dy\,dx = \int_1^5 \frac{k}{3} x y^3 \Big|_0^{1/x} dx = \int_1^5 \frac{k}{3} x^{-2}\,dx =$$

$$x \cdot \frac{1}{x^3}$$

$$-\frac{k}{3} \frac{1}{x} \Big|_1^5 = -\frac{k}{3} \cdot \frac{1}{5} + \frac{k}{3} \cdot \frac{1}{1} = \frac{k}{3} - \frac{k}{15} = \frac{4}{15} k$$

$$\bar{y} = \frac{M_x}{M} = \frac{4/15\,\cancel{k}}{\frac{\cancel{k}}{2} \ln 5} = \frac{4}{15} \cdot 2 \cdot \frac{1}{\ln 5} = \frac{8}{15 \ln 5}$$

$$\bar{x} = \frac{M_y}{M} = \frac{2\cancel{k}}{\frac{\cancel{k}}{2} \ln 5} = \frac{4}{\ln 5}$$

$$(\bar{x}, \bar{y}) = \left(\frac{4}{\ln 5}, \frac{8}{15 \ln 5}\right)$$

9. Find integrals for I_x, I_y, I_z for the cube $Q: \{(x,y,z): 0 \le x \le 2, 0 \le y \le 3, 0 \le z \le 4\}$ with density $\rho(x,y,z) = kxyz$. You do not need to integrate them. (10 points)

$$I_x = \int_0^2 \int_0^3 \int_0^4 kxyz(y^2+z^2)\, dz\, dy\, dx$$

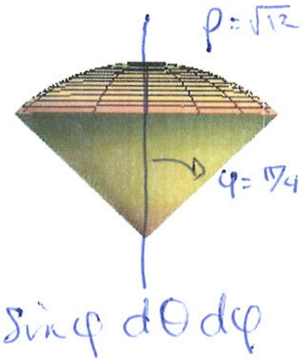
$$I_y = \int_0^2 \int_0^3 \int_0^4 kxyz\,(x^2+z^2)\, dz\, dy\, dx$$

$$I_y = \int_0^2 \int_0^3 \int_0^4 kxyz\,(x^2+y^2)\, dz\, dy\, dx$$

10. Find the volume of the solid between the cone $z = \sqrt{x^2+y^2}$ and the sphere $x^2+y^2+z^2 = 12$. (10 points)

$$\int_0^{\pi/4} \int_0^{2\pi} \int_0^{\sqrt{12}} \rho^2 \sin\varphi \, d\rho\, d\theta\, d\varphi \,.$$

$\rho\cos\varphi = \rho\sin\varphi$
$\Rightarrow \varphi = \pi/4$



$$= \int_0^{\pi/4} \int_0^{2\pi} \frac{\rho^3}{3}\Big|_0^{\sqrt{12}} \sin\varphi \, d\theta\, d\varphi = \int_0^{\pi/4} \int_0^{2\pi} \frac{\cancel{12}\sqrt{12}}{\cancel{3}} \sin\varphi \, d\theta\, d\varphi$$

$$= \int_0^{\pi/4} 4\sqrt{12}\sin\varphi \cdot \theta\Big|_0^{2\pi} d\varphi = \int_0^{\pi/4} 8\pi\sqrt{12}\sin\varphi\, d\varphi =$$

$$-8\pi\sqrt{12}\cos\varphi\Big|_0^{\pi/4} = -8\pi\underbrace{\sqrt{12}}_{2\sqrt{3}}\left(\frac{1}{\sqrt{2}} - 1\right) = \frac{16\pi\sqrt{3}\,(\sqrt{2}-1)}{\sqrt{2}}$$

$\swarrow x^2+y^2+z^2=4$ $\rho=2$

11. Convert the integral $\int_0^2\int_0^{\sqrt{4-x^2}}\int_0^{\sqrt{4-x^2-y^2}}\sqrt{x^2+y^2+z^2}\,dz\,dy\,dx$ to spherical coordinates. Evaluate the integral. (10 points)

first octant ρ

$$\int_0^{\pi/2}\int_0^{\pi/2}\int_0^2 \rho\cdot\rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi=\int_0^{\pi/2}\int_0^{\pi/2}\int_0^2\rho^3\sin\varphi\,d\rho\,d\theta\,d\varphi$$

$$=\int_0^{\pi/2}\int_0^{\pi/2}\frac{\rho^4}{4}\Big|_0^2\,d\theta\,\sin\varphi\,d\varphi=\int_0^{\pi/2}\int_0^{\pi/2}4\sin\varphi\,d\theta\,d\varphi=$$

$$\int_0^{\pi/2}4\sin\varphi\cdot\theta\Big|_0^{\pi/2}d\varphi=\int_0^{\pi/2}2\pi\sin\varphi\,d\varphi=2\pi(-\cos\varphi)\Big|_0^{\pi/2}=$$

$$-2\pi(0-1)=\boxed{2\pi}$$

12. Use the change of variables $x=\frac{1}{2}(u+v), y=\frac{1}{2}(u-v)$ to evaluate the double integral $\iint_R \ln(x+y)\,dA$ over the region bounded by the parallelogram with vertices at $(1,1),(2,2),(1,3),(0,2)$. (13 points)

$$\int_2^4\int_{-2}^0 \ln u\cdot\frac{1}{2}\,dv\,du=$$

$$\int_2^4\frac{1}{2}\ln u\cdot v\Big|_{-2}^0\,du=\int_2^4\ln u\,du$$

$p=\ln u$ $dq=du$

$dp=\frac{1}{u}du$ $q=u$

$$u\ln u-\int du=u\ln u-u\Big|_2^4=$$

$$4\ln4-4-2\ln2+2=$$

$$\ln4^4-2-\ln2^2=\ln\left(\frac{4^4}{2^2}\right)-2=$$

$$\boxed{\ln 64-2}$$

$$J=\begin{vmatrix}\frac{1}{2}&\frac{1}{2}\\\frac{1}{2}&-\frac{1}{2}\end{vmatrix}=-\frac{1}{4}-\frac{1}{4}=-\frac{1}{2}$$

$$|J|=\frac{1}{2}$$

$x+y=\frac{1}{2}(u+v)+\frac{1}{2}(u-v)$

$\Rightarrow \frac{1}{2}u+\cancel{\frac{1}{2}v}+\frac{1}{2}u-\cancel{\frac{1}{2}v}=u$

$1=\frac{1}{2}(u+v)\Rightarrow 2=u+v$

$1=\frac{1}{2}(u-v)\Rightarrow 2=u-v$

$4=2u$

$u=2\Rightarrow v=0$

$2=\frac{1}{2}(u+v)\Rightarrow 4=u+v$

$2=\frac{1}{2}(u-v)\Rightarrow 4=u-v$

$8=2u$

$u=4$

$\Rightarrow v=0$

$1=\frac{1}{2}(u+v)\Rightarrow 2=u+v$

$3=\frac{1}{2}(u-v)$ $6=u-v$

$8=2u$

$u=4$

$v=-2$

$0=\frac{1}{2}(u+v)\Rightarrow 0=u+v$

$2=\frac{1}{2}(u-v)\Rightarrow 4=u-v$

$4=2u$

$\Rightarrow u=2$

$\Rightarrow v=-2$