MATH 240, Exam #2, Summer 2017 Name KEY

Instructions: Show all work. Answers without work required to obtain the solution will not receive full credit. Some questions may contain multiple parts: be sure to answer all of them. Give exact answers unless specifically asked to estimate.

1. Find an equation for the plane through the points $A(1,0,1), B(2,0,2), C(0,3,0)$. (12 points)

$\vec{v} = \langle 1, 0, 1 \rangle$

$\vec{w} = \langle -2, 3, -2 \rangle$

$$\vec{v} \times \vec{w} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 1 \\ -2 & 3 & -2 \end{vmatrix} =$$

$(-3)\hat{i} - (-2+2)\hat{j} + (3-0)\hat{k}$

$\langle -3, 0, 3 \rangle$

$$-3(x-1) + 0(y-0) + 3(z-1) = 0$$

2. Let $z = 2x^2 - 4y^2 + 15x$, where $x = 2\cos t, y = 4\sin t$. Find $\frac{dz}{dt}$ and evaluate it at $t = \frac{\pi}{3}$. (12 points)

$\frac{\partial z}{\partial x} = z_x = 4x + 15$

$\frac{\partial z}{\partial y} = z_y = -8y$

$\frac{dx}{dt} = 2\sin t$

$\frac{dy}{dt} = -4\cos t$

$\frac{dz}{dt} = z_x \frac{dx}{dt} + z_y \frac{dy}{dt}$

$= (4x+15)(2\sin t) - 8y(-4\cos t)$

$= (4 \cdot 2\cos t + 15)(2\sin t) + 32(4\sin t)\cos t$

$= (8(1/2) + 15)(2 \frac{\sqrt{3}}{2}) + 32(4 \cdot \frac{\sqrt{3}}{2})(1/2)$

$19\sqrt{3} + 32\sqrt{3} = 51\sqrt{3}$

$\cos(\pi/3) = 1/2$

$\sin(\pi/3) = \frac{\sqrt{3}}{2}$

3. Given the implicit function $x^4 + 6x^2y - y^3 = 0$, find $\frac{dy}{dx}$. (12 points)

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{4x^3 + 12xy}{6x^2 - 3y^2} \qquad F = x^4 + 6x^2y - y^3$$

4. Find the following limits, or prove that they do not exist. (10 points each)

a. $\lim_{(x,y)\to(0,0)} \frac{x-y}{\sqrt{x}-\sqrt{y}} \cdot \frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}+\sqrt{y}} = \lim_{(x,y)\to(0,0)} \frac{\cancel{(x-y)}(\sqrt{x}+\sqrt{y})}{\cancel{x-y}} = \boxed{0}$

b. $\lim_{(x,y)\to(0,0)} \frac{2x-y^2}{2x^2+y}$ let $y = kx^2$

$$\lim_{x\to 0} \frac{2x-(kx^2)^2}{2x^2+kx^2} = \lim_{x\to 0} \frac{2x - k^2x^4}{(2+k)x^2} =$$

$$\lim_{x\to 0} \frac{\cancel{x}(2-k^2x^3)}{(2+k)x^{\cancel{2}}} = \lim_{x\to 0} \frac{2-k^2x^3}{(2+k)x} = \lim_{x\to 0}\left[\frac{2}{(2+k)x} - \frac{k^2x^{\cancel{3}\,2}}{(2+k)\cancel{x}}\right]$$

$$\lim_{x\to 0}\left[\frac{2}{(2+k)x}\right] - \lim_{x\to 0}\left[\cancel{\frac{k^2x^2}{2+k}}\right]^{\to 0} = \lim_{x\to 0} \frac{2}{(2+k)x} = \text{undefined}$$

DNE

(signs change to other sides of 0)

5. Consider the function $f(x, y) = x^3 - x^2y + 2y^2 + 6x$. Find the unit vectors that give the direction of the steepest ascent and steepest descent at the point $P(-1, -1)$. (12 points)

$\nabla f = \langle 3x^2 - 2xy + 6, -x^2 + 4y \rangle$

$= \langle 3(-1)^2 - 2(-1)(-1) + 6, -(-1)^2 + 4(-1) \rangle$

$= \langle 3 - 2 + 6, -1 - 4 \rangle = \langle 7, -5 \rangle$

$\| \langle 7, -5 \rangle \| = \sqrt{49 + 25} = \sqrt{74}$

greatest ascent $\left\langle \frac{7}{\sqrt{74}}, \frac{-5}{\sqrt{74}} \right\rangle$

greatest descent $\left\langle \frac{-7}{\sqrt{74}}, \frac{5}{\sqrt{74}} \right\rangle$

6. Find an equation of the tangent plane to the surface $10x^2 + y^2 + z^2 = 540$ at the point $(-6,6,12)$. (15 points)

$F = 10x^2 + y^2 + z^2 - 540 = 0$

$\nabla F = \langle 20x, 2y, 2z \rangle$

$= \langle 20(-6), 2(6), 2(12) \rangle =$

$= \langle -120, 12, 24 \rangle$

or $\langle -10, 1, 2 \rangle$

$-10(x+6) + (y-6) + 2(z-12) = 0$

7. Examine the function $f(x, y) = x^3 - 12xy + 8y^3$ for local maxima, minima and/or saddle points. (20 points)

$f_x = 3x^2 - 12y = 0$

$f_y = -12x + 24y^2 = 0$

$f_{xx} = 6x$

$f_{yy} = 48y$

$f_{xy} = -12$

$\frac{3x^2}{3} = \frac{12y}{3} \Rightarrow x^2 = 4y$

$\frac{12x}{12} = \frac{24y^2}{12} \Rightarrow x = 2y^2$

$(2y^2)^2 = 4y$

$\cancel{4}y^4 = \cancel{4}y \Rightarrow y^4 - y = 0$

$y(y^3 - 1) = 0 \quad y = 0, \; y = 1$

$x = 0, \quad x = 2$

$D(0,0) = 0(0) - (-12)^2 = -144 < 0$ Saddle point

$D(2,1) = (12)(48) - (-12)^2 = 576 - 144 = 432 > 0$

max or min

$f_{xx} > 0 \quad \cup \Rightarrow$ minimum

$(0,0)$

$(2,1)$

8. Use Lagrange multipliers to optimize the function $f(x, y, z) = xyz$ subject to the constraint $x + y + z = 1$. (20 points)

$\nabla f = \lambda \nabla g \qquad g = x + y + z - 1$

$yz = \lambda$

$xz = \lambda$

$xy = \lambda$

$yz = xz \Rightarrow z(y - x) = 0 \qquad z = 0$ or $y = x$

$xz = xy \Rightarrow x(z - y) = 0 \qquad x = 0$ or $y = z$

$yz = xy \Rightarrow y(z - x) = 0$

$y = 0$ or $z = x$

$0 + y + 0 = 1$

$0 + 0 + z = 1$

$x + 0 + 0 = 1$

$x + y + z = 1$

$x + x + x = 1$

$3x = 1$

$x = 1/3$

$0 + 2y = 1$

$y = 1/2$

$(0,0,1)$

$(0,1,0)$

$(1,0,0)$

$(1/3, 1/3, 1/3)$

$(0, 1/2, 1/2)$

$(1/2, 0, 1/2)$

$(1/2, 1/2, 0)$