MATH 240, Exam #1, Summer 2017 Name KEY

Instructions: Show all work. Answers without work required to obtain the solution will not receive full credit. Some questions may contain multiple parts: be sure to answer all of them. Give exact answers unless specifically asked to estimate.

1. Let $\vec{u} = \langle 3, 1, 5\rangle$ and $\vec{v} = \langle -2, 3, 1\rangle$. Find (10 points)

$$\frac{2\vec{u} + 3\vec{v}}{||\vec{u} + \vec{v}||}$$

$2\vec{u} + 3\vec{v} = 2\langle 3,1,5\rangle + 3\langle -2,3,1\rangle$
$= \langle 6,2,10\rangle + \langle -6,9,3\rangle$
$= \langle 0, 11, 13\rangle$

$\vec{u} + \vec{v} = \langle 3,1,5\rangle + \langle -2,3,1\rangle$
$= \langle 1,4,6\rangle$

$||\vec{u} + \vec{v}|| = \sqrt{1^2+4^2+6^2} = \sqrt{1+16+36}$
$= \sqrt{53}$

$$\frac{\langle 0, 11, 13\rangle}{\sqrt{53}}$$

2. Find the equation of the line that passes through the points $(3,4,1)$ and $(-5,2,7)$. Write your answer in: (8 points)
   a. Parametric form

$\vec{r}(t) = (3-8t)\hat{i} + (4-2t)\hat{j} + (1+6t)\hat{k}$

$\vec{v} = \langle -5-3, 2-4, 7-1\rangle$
$= \langle -8, -2, 6\rangle$

   b. Symmetric form

$$\frac{x-3}{-8} = \frac{y-4}{-2} = \frac{z-1}{6}$$

$x = 3-8t$
$y = 4-2t$
$z = 1+6t$

3. Find a set of parametric equations of the line through the point $(2,-3,6)$ that is parallel to the line $x=-2+3t, \quad y=8-9t, \quad z=7-t.$ (8 points)

$\vec{v}=\langle 3,-9,-1\rangle$

$x=2+3t$

$y=-3-9t$

$z=6-t$

4. Find the standard equation of the sphere that has $(5,3,-1)$ and $(3,7,9)$ as end points of the diameter. (8 points)

midpoint = center

$\left(\frac{5+3}{2}, \frac{3+7}{2}, \frac{-1+9}{2}\right)=\left(\frac{8}{2}, \frac{10}{2}, \frac{8}{2}\right)=(4,5,4)$

distance $=\sqrt{(5-3)^2+(3-7)^2+(-1-9)^2}$ diameter

$=\sqrt{2^2+(-4)^2+(-10)^2}=\sqrt{4+16+100}=\sqrt{120}=2\sqrt{30}$

radius $\frac{2\sqrt{30}}{2}=\sqrt{30}$

$(x-4)^2+(y-5)^2+(z-4)^2=30$

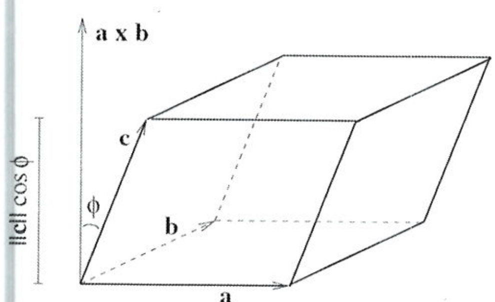
5. The projection of the vector $\vec{u}$ in the direction of $\vec{v}$ is given by $proj_{\vec{v}}\vec{u} = \left(\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v}$. Find the projection of the vector $\vec{u} = \langle 2,1,5\rangle$ in the direction of the vector $\vec{v} = \langle -9,3,2\rangle$. (8 points)

$\vec{u}\cdot\vec{v} = -18+3+10 = -5$

$\vec{v}\cdot\vec{v} = 81+9+4 = 94$

$proj_{\vec{v}}\vec{u} = \frac{-5}{94}\langle -9,3,2\rangle$

6. Consider the parallelepiped (slanted box) determined by the position vectors $\vec{a} = \langle -2,5,1\rangle, \vec{b} = \langle 1,3,-1\rangle, \vec{c} = \langle 1,4,6\rangle$. The volume is given by the triple scalar product $\left|\vec{c}\cdot(\vec{a}\times\vec{b})\right| = \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$. Find the volume of the parallelepiped. (10 points)



$\begin{vmatrix} -2 & 5 & 1 \\ 1 & 3 & -1 \\ 1 & 4 & 6 \end{vmatrix} =$

$-2\begin{vmatrix} 3 & -1 \\ 4 & 6 \end{vmatrix} - 5\begin{vmatrix} 1 & -1 \\ 1 & 6 \end{vmatrix} + 1\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} =$

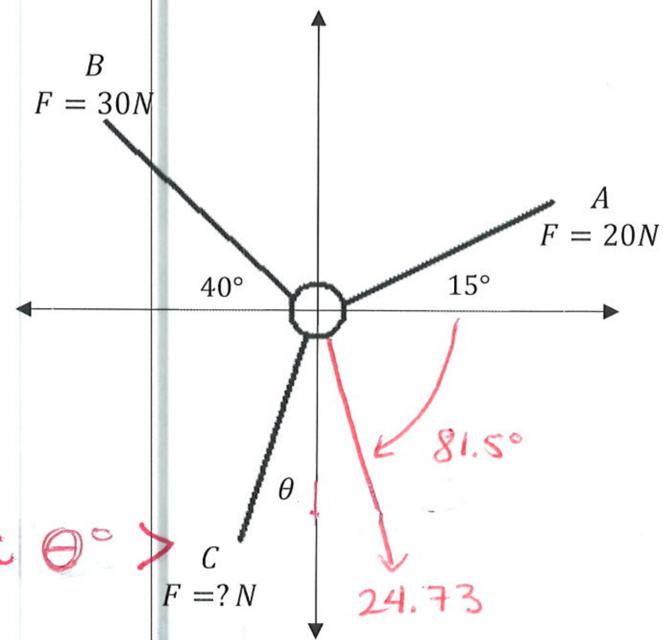
$-2(18+4) - 5(6+1) + 1(4-3) =$

$-2(22) - 5(7) + 1(1) =$

$-44 - 35 + 1 = -78$

$\boxed{78}$

7. Three people located at A, B, C pull on ropes tied to a ring. If A and B are pulling as shown, find the magnitude and direction with which C must pull so that no one moves (system is in equilibrium). You may round your final answer to 2 decimal places. (12 points)



$F_A = \langle 20\cos 15°, 20 \sin 15° \rangle$

$F_B = \langle 30 \cos 140°, 30 \sin 140° \rangle$

$F_A + F_B = \langle -3.66, 24.46 \rangle$

$F_C = \langle 3.66, -24.46 \rangle$

$\|F_C\| = 24.73$

$F_C = \langle 24.73 \cos \theta°, 24.73 \sin \theta° \rangle$

$\theta = \tan^{-1}\left(\frac{-24.46}{3.66}\right) = -81.5° = 278.5°$

or $\theta = -\cos^{-1}\left(\frac{3.66}{24.73}\right)$

8. Consider a particle traveling along a path determined by $\vec{r}(t) = 5t\hat{\imath} + 3\sin t\,\hat{\jmath} + 4\cos t\,\hat{k}$.

a. Find the velocity of the object. (3 points)

$\vec{r}'(t) = 5\hat{\imath} + 3\cos t\,\hat{\jmath} - 4\sin t\,\hat{k}$

b. What is the speed of the object? (3 points)

$\|\vec{r}'(t)\| = \sqrt{25 + 9\cos^2 t + 16\sin^2 t} = \sqrt{34 + 7\sin^2 t}$

c. What is the acceleration of the object? (3 points)

$\vec{r}''(t) = 0\hat{i} - 3\sin t\,\hat{j} - 4\cos t\,\hat{k}$

9. A volleyball is hit when it is 4 feet off the ground and 20 feet from a 6-foot-high net. It leaves the point of impact with an initial velocity of 24 ft/sec at an angle of 45° and slips by the opposing team untouched.

a. Find a vector equation for the path of the volleyball. (4 points)

$\vec{r}(t) = (v_0 t\cos\theta)\hat{i} + (-\frac{1}{2}gt^2 + v_0 t\sin\theta + h_0)\hat{j}$

$(24\cos 45°)t\hat{i} + (-16t^2 + 24\sin 45° t + 4)\hat{j}$

$12\sqrt{2}t\,\hat{i} + (-16t^2 + 12\sqrt{2}t + 4)\hat{j}$

b. How high does the volleyball go? (4 points)

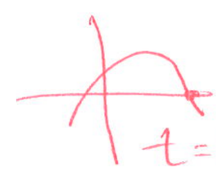
$-32t + 12\sqrt{2} = 0$

$\frac{12\sqrt{2}}{32} = t \approx .53033$

$y = 8.5$ ft

c. Find its range. (4 points)

$1.26(12\sqrt{2}) = 21.4$ feet

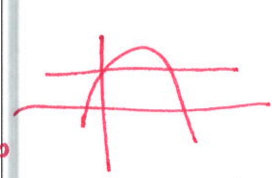


$t = 1.26$

d. When is the volleyball 8 feet above the ground? (4 points)

$t \approx .354$ seconds

$t \approx .7071$ Seconds



e. Suppose that the net is raised to 10 feet. Will the ball sail over the net? Explain by showing work. (4 points)

No, since the max height of the ball is only 8.5 feet.

10. Find the principle unit normal vector to the curve $\vec{r}(t) = 3\cos t\,\hat{\imath} + 3\sin t\,\hat{\jmath}$, at the point $t = \frac{7\pi}{6}$. (10 points)

$\vec{r}'(t) = -3\sin t\,\hat{\imath} + 3\cos t\,\hat{\jmath} \qquad \vec{T}(t) = \frac{\vec{r}'(t)}{\|r'(t)\|} = \frac{\vec{r}'(t)}{3}$

$= \vec{T}(t) = -\sin t\,\hat{\imath} + \cos t\,\hat{\jmath}$

$N(t) = \frac{\vec{T}'(t)}{\|T'(t)\|} = \frac{-\cos t\,\hat{\imath} - \sin t\,\hat{\jmath}}{1} = -\cos t\,\hat{\imath} - \sin t\,\hat{\jmath}$

$N\left(\frac{7\pi}{6}\right) = \frac{\sqrt{3}}{2}\hat{\imath} + \frac{1}{2}\hat{\jmath}$

11. Find the length of the space curve $\vec{r}(t) = 4\cos t\,\hat{\imath} - 4\sin t\,\hat{\jmath} + 8t\hat{k}$ over the interval $[0,2\pi]$. (10 points)

$\vec{r}'(t) = -4\sin t\,\hat{\imath} - 4\cos t\,\hat{\jmath} + 8\hat{k}$

$\|\vec{r}'(t)\| = \sqrt{16\sin^2 t + 16\cos^2 t + 64} = \sqrt{16+64} = \sqrt{80} = 4\sqrt{5}$

$\int_0^{2\pi} \|\vec{r}'(t)\|dt = \int_0^{2\pi} 4\sqrt{5}\,dt = \boxed{8\sqrt{5}\pi}$

12. Find the curvature of the curve given by $\vec{r}(t) = 2t\hat{\imath} + 3t^2\hat{\jmath} + t^2\hat{k}$. (10 points)

$$\vec{r}\,'(t) = 2\hat{\imath} + 6t\hat{\jmath} + 2t\hat{k}$$
$$\vec{r}\,''(t) = 0\hat{\imath} + 6\hat{\jmath} + 2\hat{k}$$

$$\vec{r}\,'(t) \times \vec{r}\,''(t) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 6t & 2t \\ 0 & 6 & 2 \end{vmatrix} =$$

$$(12t - 12t)\hat{\imath} - (4)\hat{\jmath} + (12-0)\hat{k}$$

$$0\hat{\imath} - 4\hat{\jmath} + 12\hat{k}$$

$$\|\vec{r}\,'(t) \times \vec{r}\,''(t)\| = \sqrt{(-4)^2 + 12^2} = \sqrt{16+144} = \sqrt{160} =$$
$$4\sqrt{10}$$

$$\|\vec{r}\,'(t)\| = \sqrt{2^2 + (6t)^2 + (2t)^2} = \sqrt{4 + 36t^2 + 4t^2} =$$
$$\sqrt{4 + 40t^2} = 2\sqrt{1+10t^2}$$

$$K = \frac{4\sqrt{10}}{\left(2\sqrt{1+10t^2}\right)^3} = \frac{\sqrt{10}}{2(1+10t^2)^{3/2}}$$