MAT 202, Final Exam, Part I, Summer 2017 Name KEY

Instructions: Show all work. You may **not** use a calculator on this portion of the exam. **Give exact answers** (yes, that means fractions, square roots and exponentials, and not decimals). Reduce as much as possible. Be sure to complete all parts of each question. Provide explanations where requested. When you are finished with this portion of exam, get Part II.

1. Determine if each statement is True or False. (2 points each)

a. T F Geometrically, if λ is an eigenvalue of a matrix A, and $\vec{x}$ is an eigenvector of A corresponding to λ, then multiplying $\vec{x}$ by A produces a vector $\lambda\vec{x}$ parallel to $\vec{x}$.

b. T F If A is an $n \times n$ matrix with an eigenvalue λ, then the set of all eigenvectors of λ is a subspace of R^n.

c. T F The fact that an $n \times n$ matrix A has n distinct eigenvalues does not guarantee that A is diagonalizable. it does guarantee it

d. T F When $P^{-1} = P^T$ then P contains an orthonormal basis for R^n. only if P is also square

e. T F An eigenvector for A may be the zero vector.

f. T F Similar matrices may or may not have the same eigenvalues. they must

g. T F If one of the eigenvalues of A is 3, then one of the eigenvalues of $A - 5I$ is -2.

h. T F The matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible when $ab - dc \neq 0$.. $ad - bc \neq 0$

i. T F The zero matrix is an elementary matrix. not a row operation

j. T F Cramer's Rule can only be applied when the coefficient matrix of the system is nonsingular.

k. T F The set of all ordered triples (x, y, z) of real numbers where $yz \geq 0$ with the standard operations on R^3 is a vector space. fails scalar mult. rule

l. T F Elementary row operations preserve the column space of the matrix A. preserve row space

m. T F A nonzero vector in an inner product space can have a norm of zero.

n. T F The dimension of M_{3x4} is seven. twelve

o. T F The set of vectors $\left\{ \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\}$ is linearly independent.

p. T F The dimension of a vector space is equal to the number of vectors in any basis for the space.

q. T F The orthogonal complement of R^n is the zero vector.

r. T F An orthonormal basis derived by the Gram-Schmidt orthonormalization process does depend on the order of the vectors in the basis.

s. T F If a set of vectors S is linearly independent, then S is also orthogonal.

orthogonal → independence

t. T F Two matrices that represent the same linear transformation $T: V \to W$ with respect to different bases are symmetric.

similar

2. Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}$. (8 points)

$(-2-\lambda)(2-\lambda) - 1 = 0$

$\lambda^2 - 4 - 1 = 0$

$\lambda^2 - 5 = 0$

$\lambda = -\sqrt{5}$ $\quad \lambda = \pm\sqrt{5}$

$\begin{bmatrix} -2+\sqrt{5} & 1 \\ 1 & 2+\sqrt{5} \end{bmatrix}$ $\quad \begin{matrix} x_1 = -(2+\sqrt{5})x_2 \\ x_2 = x_2 \end{matrix}$ $\quad \vec{v}_1 = \begin{bmatrix} 2+\sqrt{5} \\ -1 \end{bmatrix}$

$\lambda_2 = \sqrt{5}$ $\quad \vec{v}_2 = \begin{bmatrix} 2-\sqrt{5} \\ -1 \end{bmatrix}$

3. For the matrix $A = \begin{bmatrix} -2 & 3 \\ -1 & 2 \end{bmatrix}$ with eigenvalues $\lambda_1 = -1, \lambda_2 = 1$ with corresponding eigenvectors $\vec{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Find the similarity transformation that diagonalizes A and find D. (6 points)

$$P = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$P^{-1} = \frac{1}{3-1}\begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 3/2 \end{bmatrix}$$

4. Use the information from #3, to find e^A. (6 points)

$$e^D = \begin{bmatrix} e^{-1} & 0 \\ 0 & e \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^{-1} & 0 \\ 0 & e \end{bmatrix}\begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 3/2 \end{bmatrix} =$$

$$\begin{bmatrix} 3/e + 0 & 0 + e \\ 1/e + 0 & 0 + e \end{bmatrix}\begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 3/2 \end{bmatrix} = \begin{bmatrix} 3/2e - \frac{1}{2}e & -\frac{3}{2e} + \frac{3}{2}e \\ \frac{1}{2e} - \frac{1}{2}e & -\frac{1}{2e} + \frac{3}{2}e \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{2}e^{-1} - \frac{1}{2}e & -\frac{3}{2}e^{-1} + \frac{3}{2}e \\ \frac{1}{2}e^{-1} - \frac{1}{2}e & -\frac{1}{2}e^{-1} + \frac{3}{2}e \end{bmatrix} = e^A$$

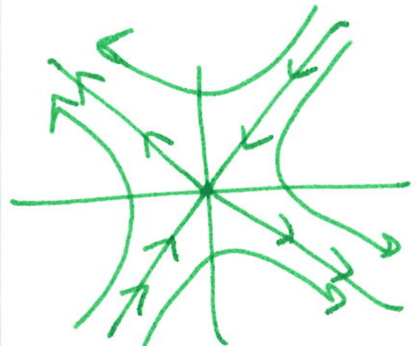
5. Solve the linear system of ODEs given by $\vec{y}' = \begin{bmatrix} 1 & -3 \\ -3 & 1 \end{bmatrix} \vec{y}$. Write the solution in standard form, and plot several sample trajectories. (10 points)

$$(1-\lambda)(1-\lambda) - 9 = 0$$
$$\lambda^2 - 2\lambda + 1 - 9 = 0$$
$$\lambda^2 - 2\lambda - 8 = 0$$
$$(\lambda - 4)(\lambda + 2) = 0$$
$$\lambda = 4, -2$$

$\lambda_1 = 4$

$\begin{bmatrix} -3 & -3 \\ -3 & -3 \end{bmatrix}$ $\quad x_1 = -x_2 \quad \vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$
$\quad\; x_2 = x_2$

$-1 \quad 1$
$1 \quad\;\; 1$



Origin is saddle point.

$\lambda_2 : -2$

$\begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}$ $\quad x_1 = x_2 \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
$\quad\; x_2 = x_2$

$1 \quad -1$

$$\vec{y} = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t}$$

6. Find the LU factorization of $A = \begin{bmatrix} 1 & 7 \\ 2 & 20 \end{bmatrix}$. (6 points)

$-2R_1 + R_2 \rightarrow R_2$ $\qquad \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} = E_1$

$\Rightarrow \begin{bmatrix} 1 & 7 \\ 0 & 6 \end{bmatrix}$

$$L = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = E_1^{-1}$$

$$U = \begin{bmatrix} 1 & 7 \\ 0 & 6 \end{bmatrix}$$

7. Let M be a 9×5 matrix. If M has 5 pivots, find the following: (3 points each)
   a. $\dim Col\ A$

      5

   b. $rank\ A$

      5

   c. $\dim Nul\ A$

      0

   d. $nullity\ of\ A$

      0

   e. $Col\ A$ is a subspace of R^m. $m =?$ 9

   f. $Nul\ A$ is a subspace of R^n. $n =?$ 5

8. Use properties of determinants and $\det(A) = -2, \det(B) = 4$, to find the following, given that both A and B are 4×4 matrices. (3 points each)
   a. $\det(B^{-1})$ $= \frac{1}{4}$

   b. $\det(A^{-1}B^3)$ $= -\frac{1}{2}(4^3) = -32$

   c. $\det(5A)$ $= 5^4(-2) = -1250$

   d. $\det(A^{-1}BA)$ $= -\frac{1}{2}(4)(-2) = 4$

   e. $\det(-B^T)$ $= (-1)^4(4) = 4$

9. Use any method to find the determinant of $\begin{bmatrix} 1 & 0 & -1 & 0 & -1 \\ -1 & -1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & -2 \\ 0 & 1 & 1 & 1 & 0 \\ -1 & 2 & 1 & -1 & 0 \end{bmatrix}$. (7 points)

$R_4 + R_5 \to R_5$

$$\begin{vmatrix} 1 & 0 & -1 & 0 & -1 \\ -1 & -1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & -2 \\ 0 & 1 & 1 & 1 & 0 \\ -1 & 3 & 2 & 0 & 0 \end{vmatrix} \Rightarrow \begin{vmatrix} 1 & 0 & -1 & -1 \\ -1 & -1 & 0 & -1 \\ 1 & 0 & 0 & -2 \\ -1 & 3 & 2 & 0 \end{vmatrix}$$

$2R_1 + R_4 \to R_4$

$$\begin{vmatrix} 1 & 0 & -1 & -1 \\ -1 & -1 & 0 & -1 \\ 1 & 0 & 0 & -2 \\ 1 & 3 & 0 & -2 \end{vmatrix} \to -1 \begin{vmatrix} -1 & -1 & -1 \\ 1 & 0 & -2 \\ 1 & 3 & -2 \end{vmatrix} \to \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & 3 & -2 \end{vmatrix}$$

$-3R_1 + R_3 \to R_3$

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ -2 & 0 & -5 \end{vmatrix} \to -1 \begin{vmatrix} 1 & 2 \\ -2 & -5 \end{vmatrix} \to -1(-5+4) = -1(-1) = 1$$

10. Determine if the following transformation is linear or nonlinear. If it is linear, prove it. If it is not, find a counterexample. (6 points)

$$T: P_3 \to P_3, T\big(p(x)\big) = p'''(x) - 9p'(x) + p(x)$$

$T(0) = 0''' - 9\,0' + 0 = 0$ ✓

$T(p(x) + g(x)) = (p(x) + g(x))''' - 9(p(x) + g(x))' + (p(x) + g(x))$

$= p'''(x) + g'''(x) - 9p'(x)' - 9g(x) + p(x) + g(x) =$

$T(p(x)) + T(g(x))$ ✓

$T(kp(x)) = (kp(x))''' + -9(k\,p(x))' + kp(x) =$

$kp''' - 9kp'(x) + kp(x) =$

$kT(p(x))$ ✓

it is linear

MAT 202, Final Exam, Part II, Summer 2017 Name ____________________

Instructions: Show all work. You **may** use a calculator on this portion of the exam. To show work on calculator problems, show the commands you used, and the resulting matrices. **Give exact answers** (yes, that means fractions, square roots and exponentials, and not decimals) unless specifically directed to give a decimal answer. This will require some operations to be done by hand even if not specifically directed to. Be sure to complete all parts of each question. Provide explanations where requested.

1. Given $A = \begin{bmatrix} -2 & 13 \\ -1 & 4 \end{bmatrix}$, and $\lambda = 1 \pm 2i$, and $\vec{v} = \begin{bmatrix} 3 \pm 2i \\ 1 \end{bmatrix}$, find a similarity transformation that will create a scaled rotation matrix similar to A. Find the angle of rotation. (7 points)

$P = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$

$\theta = \tan^{-1}\left(\frac{2}{1}\right) = 1.107$ radians

$63.4°$

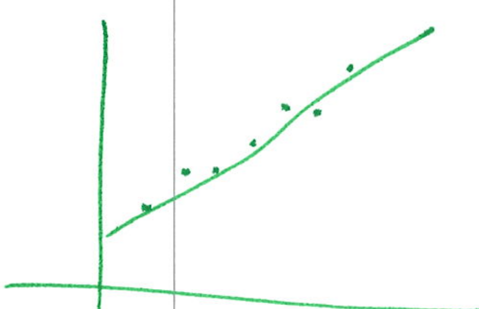
2. The table below shows world population in billions. Create a scatterplot and find a linear regression equation that best fits the data. (8 points)

Year	1950	1960	1970	1980	1990	2000
Population	2.56	3.04	3.71	4.46	5.28	6.08

$\begin{bmatrix} 1 & 1950 \\ 1 & 1960 \\ 1 & 1970 \\ 1 & 1980 \\ 1 & 1990 \\ 1 & 2000 \end{bmatrix} = A \quad \vec{y} = \begin{bmatrix} 2.56 \\ 3.04 \\ 3.71 \\ 4.46 \\ 5.28 \\ 6.08 \end{bmatrix} \quad (A^TA)^{-1}A^T\vec{y} = \vec{b}$

$= \begin{bmatrix} -137.28 \\ .0716 \end{bmatrix}$

$y = .0716x - 137.28$



3. Encode the message THEY THAT SOW THE WIND WILL REAP THE WHIRLWIND with the matrix $A = \begin{bmatrix} 4 & 2 & -1 \\ 3 & 1 & 1 \\ 0 & -2 & 1 \end{bmatrix}$ using 0 as a space and the corresponding position (number) in the alphabet for letters. (8 points)

$$\begin{bmatrix} 20 & 8 & 5 \\ 25 & 0 & 20 \\ 8 & 1 & 20 \\ 0 & 19 & 15 \\ 23 & 0 & 20 \\ 8 & 5 & 0 \\ 23 & 9 & 14 \\ 4 & 0 & 23 \\ 9 & 12 & 12 \\ 0 & 18 & 5 \\ 1 & 16 & 0 \\ 20 & 8 & 5 \\ 0 & 23 & 8 \\ 9 & 18 & 12 \end{bmatrix} = A \qquad \boxed{\begin{matrix} 23 & 9 & 14 \\ 4 & 0 & 0 \end{matrix}}$$

$\Rightarrow$

104	38	-7	100	10	-5
35	-23	13	57	-11	34
92	6	-3	47	21	-3
119	27	0	16	-38	19
72	6	15	54	8	23
52	18	15	104	38	-7
69	7	31	90	12	21
119	27	0	16	8	-4

4. Give an example of a matrix that is orthogonally diagonalizable. (It should be at least 3×3, and not already diagonal.) (5 points)

$$\begin{bmatrix} 1 & 4 & 5 \\ 4 & 2 & 6 \\ 5 & 6 & 3 \end{bmatrix}$$

5. Determine the type of conic sections given by the equation $12x^2 + 8xy + 12y^2 - 8 = 0$. (6 points)

$$\begin{bmatrix} 12 & 4 \\ 4 & 12 \end{bmatrix}$$

$(12-\lambda)(12-\lambda)-16=0$

$\lambda^2-24\lambda+144-16=0$

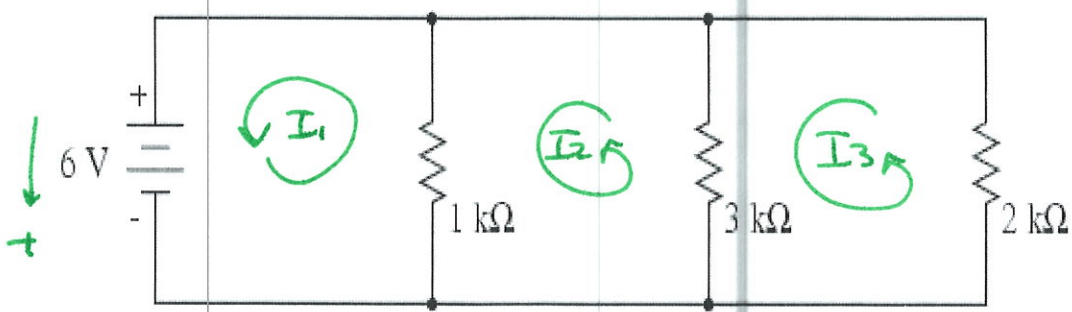
$\lambda^2-24\lambda+128=0$

$(\lambda-8)(\lambda-16)=0$

$\lambda=8, \lambda=16$

ellipse

6. Set up and solve the loop circuit diagram below. Round your values for the currents to three significant digits. (8 points)



$$1000 I_1 - 1000 I_2 = 6$$
$$-1000 I_1 + 4000 I_2 - 3000 I_3 = 0$$
$$-3000 I_2 + 5000 I_3 = 0$$

$$\left[\begin{array}{ccc|c} 1000 & -1000 & 0 & 6 \\ -1000 & 4000 & -3000 & 0 \\ 0 & -3000 & 5000 & 0 \end{array}\right] \Rightarrow \vec{I} = \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} .011 \\ .005 \\ .003 \end{bmatrix}$$

7. Write a system of equations to determine if $M = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$ can be written as a linear combination of $M_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $M_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $M_3 = \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}$. You don't need to solve the system. (6 points)

$$a\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$$

$$a = 2$$
$$b - c = 1$$
$$b = 3$$
$$-a + c = 4$$

8. If $\vec{v} = \begin{bmatrix} -2 \\ 2 \\ 3 \end{bmatrix}$ in the standard basis and $B = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$ and $C = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} \right\}$, find the following: (3 points each)

a. The transition matrix P_C

$$P_C = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & 1 & 0 \end{bmatrix}$$

b. The transition matrix $P_{B \leftarrow C}$

$$P_B^{-1} P_C = \begin{bmatrix} -2 & 1 & -1 \\ 1 & 0 & 1 \\ 0 & 3 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -2/3 & -2/3 & 2 \\ 1/3 & 1/3 & 0 \\ 2/3 & 5/3 & -3 \end{bmatrix}$$

c. $[\vec{v}]_C$

$$P_C^{-1} \vec{v} = \begin{bmatrix} -1/2 \\ 7/2 \\ 3/2 \end{bmatrix}$$

d. $[\vec{v}]_B$

$$P_B^{-1} \vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

9. Find the QR factorization for $A = \begin{bmatrix} 1 & 4 \\ 3 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$. (6 points)

$$\begin{bmatrix} 4 \\ 0 \\ 1 \\ 2 \end{bmatrix} - \frac{4+0+1+2}{1+9+1+1} \begin{bmatrix} 1 \\ 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 1 \\ 2 \end{bmatrix} - \frac{7}{12} \begin{bmatrix} 1 \\ 3 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 41/12 \\ -21/12 \\ 5/12 \\ 17/12 \end{bmatrix} \rightarrow \begin{bmatrix} 41 \\ -21 \\ 5 \\ 17 \end{bmatrix}$$

$\| \, \| = 2\sqrt{3}$ $\quad$ $\| \, \| = 2\sqrt{609}$

$$Q = \begin{bmatrix} \frac{1}{2\sqrt{3}} & \frac{41}{2\sqrt{609}} \\ \frac{3}{2\sqrt{3}} & \frac{-21}{2\sqrt{609}} \\ \frac{1}{2\sqrt{3}} & \frac{5}{2\sqrt{609}} \\ \frac{1}{2\sqrt{3}} & \frac{17}{2\sqrt{609}} \end{bmatrix}$$

$$R = Q^T A = \begin{bmatrix} \frac{1}{2\sqrt{3}} & \frac{3}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} \\ \frac{41}{2\sqrt{609}} & \frac{-21}{2\sqrt{609}} & \frac{5}{2\sqrt{609}} & \frac{17}{2\sqrt{609}} \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 3 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2\sqrt{3}} + \frac{9}{2\sqrt{3}} + \frac{1}{2\sqrt{3}} + \frac{1}{2\sqrt{3}} & \frac{4}{2\sqrt{3}} + 0 + \frac{1}{2\sqrt{3}} + \frac{2}{2\sqrt{3}} \\ \frac{41}{2\sqrt{609}} - \frac{63}{2\sqrt{609}} + \frac{5}{2\sqrt{609}} + \frac{17}{2\sqrt{609}} & \frac{164}{2\sqrt{609}} + 0 + \frac{5}{2\sqrt{609}} + \frac{34}{2\sqrt{609}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{6}{\sqrt{3}} & 4/\sqrt{3} \\ 0 & \frac{203}{2\sqrt{609}} \end{bmatrix} = R$$

10. Consider the linear transformation $T\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right) = \begin{bmatrix} a+b+d \\ b-c-d \\ a+b+d \\ c-a \end{bmatrix}$. (4 points each)

a. Write the matrix of the transformation.

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & -1 \\ 1 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{bmatrix}$$

b. What is the domain of the transformation? $\mathbb{R}^4$

c. What is the codomain of the transformation? $\mathbb{R}^4$

d. What is the dimension of the range? rref $\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ 3

e. Find a basis for the kernel.

$x_1 = -x_4$
$x_2 = 0$
$x_3 = -x_4$
$x_4 = x_4$

$$\begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

f. Is the transformation one-to-one?

no

g. Is the transformation onto?

no

h. Is the transformation an isomorphism? Why or why not?

must be both one-to-one and onto.