MAT 202, Exam #3, Part I, Summer 2017 Name KEY

Instructions: Show all work. You may **not** use a calculator on this portion of the exam. **Give exact answers** (yes, that means fractions, square roots and exponentials, and not decimals). Reduce as much as possible. Be sure to complete all parts of each question. Provide explanations where requested. When you are finished with this portion of exam, get Part II.

1. Determine if each statement is True or False. (1 point each)

a. T F If $\vec{v}$ is a nonzero vector in R^n, then the unit vector in the direction of $\vec{v}$ is $\hat{u} = \frac{\|\vec{v}\|}{\vec{v}}$. $\frac{\vec{v}}{\|\vec{v}\|}$

b. T F If $\vec{u} \cdot \vec{v} < 0$, then the angle θ between $\vec{u}$ and $\vec{v}$ is acute. obtuse

c. T F The dot product is the only inner product that can be defined on R^n.

d. T F A nonzero vector in an inner product space can have a norm of zero. only $\vec{0}$ can be 0 norm

e. T F An orthonormal basis derived by the Gram-Schmidt orthonormalization process does not depend on the order of the vectors in the basis. it does

f. T F If a set of vectors S is linearly independent, then S is also orthogonal. the reverse is true

g. T F The orthogonal complement of R^n is the empty set. $\vec{0}$

h. T F For polynomials, the differential operator D_x is a linear transformation from $P_n \to P_{n+1}$. P_{n-1}

i. T F The vector spaces R^5 and P_4 are isomorphic to each other.

j. T F The function $f(x) = x^3$ is a linear transformation from $R \to R$.

k. T F The nullity is the number of free variables in a matrix.

l. T F The set of all vectors mapped from a vector space V into another vector space W by a linear transformation T is the kernel of T. range

m. T F A linear transformation T from V to W is one-to-one when the preimage of every $\vec{w}$ in the range consists of a single vector $\vec{v}$.

n. T F All linear transformations T have a unique inverse T^{-1}.

o. T F Two matrices that represent the same linear transformation $T: V \to W$ with respect to different bases are not necessarily similar.

p. T F The matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ is one-to-one. it's onto

q. T F The matrix $A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ represents a horizontal shear.

2. Determine if the following transformations are linear or nonlinear. If they are linear, prove it. If they are not, find a counterexample. (5 points each)

a. $T: R^3 \to R^3, T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} \cos x \\ \sin y \\ \sin x + \sin z \end{bmatrix}$.

nonlinear $T(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

b. $T: P_2 \to P_2, T(p(x)) = p''(x) - 2p'(x) + p(x)$

linear $T(p(x)+q(x)) = (p(x)+q(x))'' - 2(p(x)+q(x)) + (p(x)+q(x)) = p''(x) - 2p(x) + p(x) + q''(x) - 2q(x) + q(x) = T(p(x)) + T(q(x))$

$T(0) = 0'' + 2(0)' + 0 = 0$

$kT(p(x)) = k(p''(x) - 2p'(x) + p(x)) = T(kp(x)) = kp''(x) - 2kp'(x) + kp(x)$.

3. Write the matrix of the transformation for $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x - y + 3z \\ x + y + z \\ -x + 3y - 5z \end{bmatrix}$. (3 points)

$A = \begin{bmatrix} 1 & -1 & 3 \\ 1 & 1 & 1 \\ -1 & 3 & -5 \end{bmatrix}$

4. Find an isomorphism from $\boldsymbol{M}_{2\times 2}$ onto $\boldsymbol{P}_3$. (4 points)

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \to ax^3 + bx^2 + cx + d$

5. Use the matrix $P = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$ to find a matrix similar to $A = \begin{bmatrix} -2 & -1 \\ 3 & 2 \end{bmatrix}$. (5 points)

$$P^{-1}AP = \begin{bmatrix} 1/3 & 2/3 \\ 1/3 & -1/3 \end{bmatrix}\begin{bmatrix} -2 & -1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 7/3 & 5/3 \\ -8/3 & -7/3 \end{bmatrix}$$

$$PAP^{-1} = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} -2 & -1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1/3 & 2/3 \\ 1/3 & -1/3 \end{bmatrix} = \begin{bmatrix} 7/3 & 5/3 \\ -8/3 & -7/3 \end{bmatrix}$$

6. Given the vector $\vec{u} = \begin{bmatrix} -3 \\ -2 \\ 3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -1 \\ -1 \\ -3 \end{bmatrix}$, find the following: (2 points each)

a. $||\vec{u}||$

$$\sqrt{9+4+9} = \sqrt{22}$$

b. A unit vector in the direction of v

$$||\vec{v}|| = \sqrt{1+1+9} = \sqrt{11} \qquad \hat{v} = \begin{bmatrix} -1/\sqrt{11} \\ -1/\sqrt{11} \\ -3/\sqrt{11} \end{bmatrix}$$

c. $||\vec{u} - \vec{v}||$

$$\vec{u} - \vec{v} = \begin{bmatrix} -2 \\ -1 \\ 6 \end{bmatrix} \qquad ||\vec{u} - \vec{v}|| = \sqrt{4+1+36} = \sqrt{41}$$

d. $\vec{u} \cdot \vec{v}$

$$3+2-9 = -4$$

e. Are $\vec{u}$ and $\vec{v}$ orthogonal? If not, is the angle between the vectors acute or obtuse?

not orthogonal , obtuse

7. Find the projection of $\vec{u} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$ onto $\vec{v} = \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix}$. (4 points)

$$\frac{2-3+4}{4+9+16}\begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix} = \frac{3}{29}\begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 6/29 \\ 9/29 \\ -12/29 \end{bmatrix}$$

8. Given the inner product $\langle f, g\rangle = \int_0^1 f(x)g(x)dx$, find the following for $f(x) = 3x - 2, g(x) = x^2 + 1$. (6 points each)

a. $\langle f, g\rangle$

$$\int_0^1 (3x-2)(x^2+1)\,dx = \int_0^1 3x^3-2x^2+3x-2\,dx$$

$$\frac{3}{4}x^4 - \frac{2}{3}x^3 + \frac{3}{2}x^2 - 2x \Big|_0^1 = \frac{3}{4} - \frac{2}{3} + \frac{3}{2} - 2 = \frac{-5}{12}$$

b. $||g||$

$$\sqrt{\int_0^1 (x^2+1)^2 dx} = \sqrt{\int_0^1 x^4+2x^2+1\,dx} = \sqrt{\frac{1}{5}x^5 + \frac{2}{3}x^3 + x \Big|_0^1} =$$

$$\sqrt{\frac{1}{5} + \frac{2}{3} + 1} = \sqrt{\frac{28}{15}}$$

MAT 202, Exam #3, Part II, Summer 2017 Name ______________________

Instructions: Show all work. You **may** use a calculator on this portion of the exam. To show work on calculator problems, show the commands you used, and the resulting matrices. **Give exact answers** (yes, that means fractions, square roots and exponentials, and not decimals) unless specifically directed to give a decimal answer. This will require some operations to be done by hand even if not specifically directed to. Be sure to complete all parts of each question. Provide explanations where requested.

1. Consider the linear transformation $T\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right) = \begin{bmatrix} a+b \\ b-c \\ a+d \end{bmatrix}$. (3 points each)

   a. Write the matrix of the transformation.

   $$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

   b. What is the domain of the transformation? $\mathbb{R}^4$

   c. What is the codomain of the transformation? $\mathbb{R}^3$

   d. What is the dimension of the range? $\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ $\mathbb{R}^3$, 3

   e. Find a basis for the kernel.

   $x_1 = -x_4$
   $x_2 = 0$
   $x_3 = 0$
   $x_4 = x_4$

   $$\left\{ \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

   f. Is the transformation one-to-one?

   no

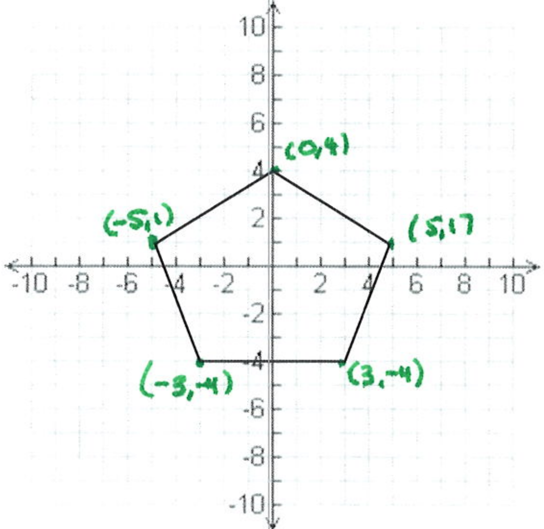
   g. Is the transformation onto?

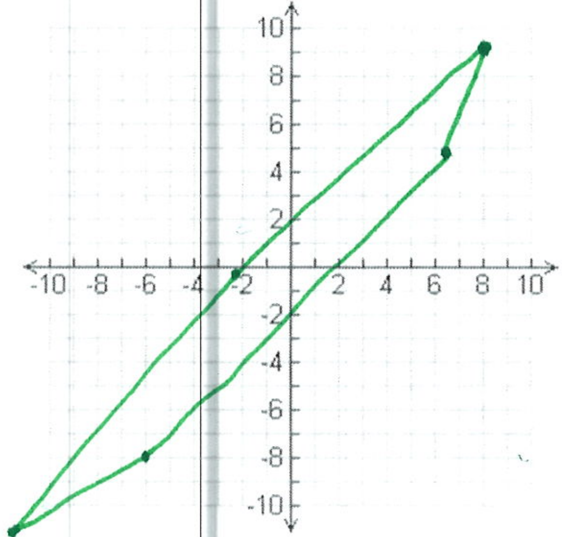
   yes

   h. Is the transformation an isomorphism? Why or why not?

   no needs to be both one-to-one and onto

2. Find a linear transformation that reflects through the y-axis, followed by a horizontal shear by a factor of 3, followed by a rotation of 30°. Then apply the transformation to the object below. (6 points)





$$\begin{bmatrix}\sqrt{3}/2 & -1/2\\ 1/2 & \sqrt{3}/2\end{bmatrix}\begin{bmatrix}1 & 3\\ 0 & 1\end{bmatrix}\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix} = \begin{bmatrix}-\sqrt{3}/2 & 3\sqrt{3}/2-1/2\\ -1/2 & \frac{3}{2}+\frac{\sqrt{3}}{2}\end{bmatrix} = T$$

$$T\begin{bmatrix}5\\1\end{bmatrix} = \begin{bmatrix}-5\sqrt{3}/2+3\sqrt{3}/2-\frac{1}{2}\\ -5/2+3/2+\sqrt{3}/2\end{bmatrix} \approx \begin{bmatrix}-2.2\\ -.1\end{bmatrix} \quad T\begin{bmatrix}3\\-4\end{bmatrix} = \begin{bmatrix}-3\sqrt{3}/2-12\sqrt{3}/2+4/2\\ -3/2-12/2-4\sqrt{3}/2\end{bmatrix} \approx \begin{bmatrix}-11\\ -11\end{bmatrix}$$

$$T\begin{bmatrix}0\\4\end{bmatrix} = \begin{bmatrix}0+12\sqrt{3}/2-4/2\\ 0+12/2+4\sqrt{3}/2\end{bmatrix} \approx \begin{bmatrix}8.4\\ 9.5\end{bmatrix}$$

$$T\begin{bmatrix}-5\\1\end{bmatrix} = \begin{bmatrix}5\sqrt{3}/2+3\sqrt{3}/2-1/2\\ 5/2+3/2+\sqrt{3}/2\end{bmatrix} \approx \begin{bmatrix}6.4\\ 4.9\end{bmatrix}$$

$$T\begin{bmatrix}-3\\-4\end{bmatrix} = \begin{bmatrix}3\sqrt{3}/2-12\sqrt{3}/2+4/2\\ 3/2-12/2-4\sqrt{3}/2\end{bmatrix} \approx \begin{bmatrix}-5.8\\ 8.0\end{bmatrix}$$

3. Find a vector orthogonal to both $\begin{bmatrix}2\\-3\\1\\1\\0\end{bmatrix}$ and $\begin{bmatrix}2\\3\\1\\2\\-1\end{bmatrix}$. (3 points)

$$\begin{bmatrix}2 & -3 & 1 & 1 & 0\\ 2 & 3 & 1 & 2 & -1\end{bmatrix} \Rightarrow \begin{bmatrix}1 & 0 & 1/2 & 3/4 & -1/4\\ 0 & 1 & 0 & 1/6 & -1/6\end{bmatrix} \qquad \begin{bmatrix}-1/2\\0\\1\\0\\0\end{bmatrix} \to \begin{bmatrix}-1\\0\\2\\0\\0\end{bmatrix}$$

$$\text{or } \begin{bmatrix}-3/4\\-1/6\\0\\1\\0\end{bmatrix} \to \begin{bmatrix}-9\\-2\\0\\12\\0\end{bmatrix} \text{ or } \begin{bmatrix}1/4\\1/6\\0\\0\\1\end{bmatrix} \to \begin{bmatrix}3\\2\\0\\0\\12\end{bmatrix}$$

or any linear combination of these

4. Use Gram-Schmidt to find an orthogonal basis spanning $S = \left\{ \begin{bmatrix}1\\1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\-1\\-1\\-1\end{bmatrix}, \begin{bmatrix}-1\\3\\1\\-1\end{bmatrix} \right\}$. (6 points)

$\vec{v}_1, \vec{v}_2, \vec{v}_3$

$\vec{u}_1 = \vec{v}_1$

$$\vec{u}_2 = \begin{bmatrix}1\\-1\\-1\\-1\end{bmatrix} - \frac{1-1-1+0}{1+1+1+0}\begin{bmatrix}1\\1\\1\\0\end{bmatrix} = \begin{bmatrix}1\\-1\\-1\\-1\end{bmatrix} + \frac{1}{3}\begin{bmatrix}1\\1\\1\\0\end{bmatrix} = \begin{bmatrix}4/3\\-2/3\\-2/3\\-1\end{bmatrix} \Rightarrow \begin{bmatrix}4\\-2\\-2\\-3\end{bmatrix} \; \vec{u}_2$$

$$\vec{u}_3 = \begin{bmatrix}-1\\3\\1\\-1\end{bmatrix} - \frac{-1+3+1+0}{1+1+1}\begin{bmatrix}1\\1\\1\\0\end{bmatrix} - \frac{-4-6-2+3}{16+4+4+9}\begin{bmatrix}4\\-2\\-2\\-3\end{bmatrix} =$$

$$\begin{bmatrix}-1\\3\\1\\-1\end{bmatrix} - \frac{3}{3}\begin{bmatrix}1\\1\\1\\0\end{bmatrix} + \frac{9}{33}\begin{bmatrix}4\\-2\\-2\\-3\end{bmatrix} = \begin{bmatrix}-1\\3\\1\\-1\end{bmatrix} - \begin{bmatrix}1\\1\\1\\0\end{bmatrix} + \frac{3}{11}\begin{bmatrix}4\\-2\\-2\\-3\end{bmatrix} = \begin{bmatrix}-10/11\\16/11\\-6/11\\-20/11\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}-10\\16\\-6\\-20\end{bmatrix} \text{ or } \begin{bmatrix}-5\\8\\-3\\-10\end{bmatrix}$$

$$\left\{ \begin{bmatrix}1\\1\\1\\0\end{bmatrix}, \begin{bmatrix}4\\-2\\-2\\-3\end{bmatrix}, \begin{bmatrix}-10\\16\\-6\\20\end{bmatrix} \right\}$$

5. Let $W = \left\{ \begin{bmatrix}3\\0\\-1\\2\end{bmatrix}, \begin{bmatrix}0\\2\\2\\1\end{bmatrix} \right\}$. Find an orthogonal basis for $W^{\perp}$. (5 points)

$$\begin{bmatrix}3 & 0 & -1 & 2\\0 & 2 & 2 & 1\end{bmatrix} \Rightarrow \begin{bmatrix}1 & 0 & -1/3 & 2/3\\0 & 1 & 1 & 1/2\end{bmatrix}$$

$x_1 = 1/3 x_3 - 2/3 x_4$
$x_2 = -x_3 - 1/2 x_4$
$x_3 = x_3$
$x_4 = x_4$

$$\begin{bmatrix}1/3\\-1\\1\\0\end{bmatrix} \Rightarrow \begin{bmatrix}1\\-3\\3\\0\end{bmatrix} \;\; 3/\text{or} \;\; \begin{bmatrix}-2/3\\-1/2\\0\\1\end{bmatrix}$$

$$\begin{bmatrix}3 & 0 & -1 & 2\\0 & 2 & 2 & 1\\1 & -3 & 3 & 0\end{bmatrix} \Rightarrow \begin{bmatrix}1 & 0 & 0 & 27/38\\0 & 1 & 0 & 7/19\\0 & 0 & 1 & 5/38\end{bmatrix}$$

$x_1 = -27/38 x_4$
$x_2 = -7/19 x_4$
$x_3 = -5/38 x_4$
$x_4 = 1$

$$\begin{bmatrix}-27\\-14\\-5\\38\end{bmatrix}$$

$$W^{\perp} = \left\{ \begin{bmatrix}1\\-3\\3\\0\end{bmatrix}, \begin{bmatrix}-27\\-14\\-5\\38\end{bmatrix} \right\}$$

6. Find the QR factorization for $A = \begin{bmatrix} 1 & 4 \\ 3 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$. (4 points)

$\vec{u_1}$ $\vec{u_2}$

$\sqrt{1+9+1+1} = \sqrt{12} = 2\sqrt{3}$

Q

$$\begin{bmatrix} 4 \\ 0 \\ 1 \\ 2 \end{bmatrix} - \frac{4+0+1+2}{1+9+1+1}\begin{bmatrix} 1 \\ 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 1 \\ 2 \end{bmatrix} - \frac{7}{12}\begin{bmatrix} 1 \\ 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 41/12 \\ -21/12 \\ 5/12 \\ 17/12 \end{bmatrix} \Rightarrow \begin{bmatrix} 41 \\ -21 \\ 5 \\ 17 \end{bmatrix}$$

$$\sqrt{41^2 + 21^2 + 5^2 + 17^2} = \sqrt{2436} = 2\sqrt{609}$$

$$Q = \begin{bmatrix} \frac{1}{2\sqrt{3}} & \frac{41}{2\sqrt{609}} \\ \frac{3}{2\sqrt{3}} & \frac{-21}{2\sqrt{609}} \\ \frac{1}{2\sqrt{3}} & \frac{5}{2\sqrt{609}} \\ \frac{1}{2\sqrt{3}} & \frac{17}{2\sqrt{609}} \end{bmatrix}$$

$$R = Q^TA = \begin{bmatrix} \frac{1}{2\sqrt{3}} & \frac{3}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} \\ \frac{41}{2\sqrt{609}} & \frac{-21}{2\sqrt{609}} & \frac{5}{2\sqrt{609}} & \frac{17}{2\sqrt{609}} \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 3 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} =$$

$$\begin{bmatrix} \frac{1}{2\sqrt{3}} + \frac{9}{2\sqrt{3}} + \frac{1}{2\sqrt{3}} + \frac{1}{2\sqrt{3}} & \frac{4}{2\sqrt{3}} + 0 + \frac{1}{2\sqrt{3}} + \frac{2}{2\sqrt{3}} \\ \frac{41}{2\sqrt{609}} - \frac{63}{2\sqrt{609}} + \frac{5}{2\sqrt{609}} + \frac{17}{2\sqrt{609}} & \frac{164}{2\sqrt{609}} + 0 + \frac{5}{2\sqrt{609}} + \frac{34}{2\sqrt{609}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{6}{\sqrt{3}} & \frac{7}{2\sqrt{3}} \\ 0 & \frac{203}{2\sqrt{609}} \end{bmatrix}$$