MAT 202, Exam #2, Part I, Summer 2017 Name KEY

Instructions: Show all work. You may **not** use a calculator on this portion of the exam. **Give exact answers** (yes, that means fractions, square roots and exponentials, and not decimals). Reduce as much as possible. Be sure to complete all parts of each question. Provide explanations where requested. When you are finished with this portion of exam, get Part II.

1. Determine if each statement is True or False. (1 point each)
   a. T F To find the determinant of a matrix, expand by cofactors in any row or column.
   b. T F If two rows of a matrix are equal, then the determinant of the matrix is zero.
   c. T F Adding a multiple of one column of a matrix to another column changes only the sign of the determinant. can change more or not at all
   d. T F If the determinant of an $n \times n$ matrix A is nonzero, then $A\vec{x} = \vec{0}$ has only the trivial solution.
   e. T F Cramer's Rule can only be applied when the coefficient matrix of the system is singular. must be nonsingular
   f. T F The cofactor C_{22} of a given matrix is always a positive number. we add in this position but cofactor can be negative
   g. T F If A is a square matrix, then the matrix of cofactors of A is called the adjoint of A.
   h. T F Three points $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ are collinear when the determinant of the matrix that has the coordinates as entries in the first two columns and 1's in the third column is nonzero. zero nonzero means it forms a triangle
   i. T F To subtract two vectors in R^n, subtract their corresponding components.
   j. T F The zero vector $\vec{0}$ in R^n is defined as the additive inverse of a vector. is $-\vec{a}$
   k. T F The set of all 3×3 matrices of the form $\begin{bmatrix} 0 & a & b \\ c & 0 & d \\ e & f & 0 \end{bmatrix}$ forms a vector space.
   l. T F The set of all ordered triples (x, y, z) of real numbers where $y \geq 0$ with the standard operations on R^3 is a vector space. will fail scalar multiple test
   m. T F To show that a set is not a vector space, it is sufficient to show that just one axiom is not satisfied.
   n. T F Every vector space V contains two proper subspaces that are the zero subspace, and itself.

o. T F — The dimension of $M_{3\times 3}$ is six. it's nine

p. T F — The set of vectors $\left\{\begin{bmatrix}4\\5\\1\end{bmatrix}, \begin{bmatrix}-1\\1\\0\end{bmatrix}\right\}$ spans R^3. only 2D subspace

q. T F — If $\dim(V) = n$ then there exists a set of $n+1$ vectors in V that will span V. yes, but need only n

r. T F — The nullity of A is the dimension of the nullspace of A.

s. T F — If an $m\times n$ matrix A is row-equivalent to an $m\times n$ matrix B, then the row space of A is equivalent to the row space of B. Column space not the same

t. T F — The coordinate vector of $p(x) = 5x^2 + x - 3$ relative to the standard basis for P_2 is $\begin{bmatrix}-3\\1\\5\end{bmatrix}$.

u. T F — Elementary row operations preserve the column space of the matrix A.

v. T F — The set of points on the line $x + y = 0$ is a subspace of R^2.

2. Find the determinant of the matrix $\begin{bmatrix}-1&1&1&2\\3&-2&0&-1\\0&1&0&1\\3&3&3&3\end{bmatrix}$ by the cofactor method. (6 points)

$$-1\begin{vmatrix}-1&1&2\\3&0&-1\\3&3&3\end{vmatrix} - 1\begin{vmatrix}-1&1&1\\3&-2&0\\3&3&3\end{vmatrix}$$

$$-\left[-1\begin{vmatrix}3&-1\\3&3\end{vmatrix} - 3\begin{vmatrix}-1&2\\3&-1\end{vmatrix}\right] - \left[1\begin{vmatrix}3&-2\\3&3\end{vmatrix} + 3\begin{vmatrix}-1&1\\3&-2\end{vmatrix}\right]$$

$$-\left[-1(9+3) - 3(1-6)\right] - \left[(9+6) + 3(2-3)\right]$$

$$-\left[-12+15\right] - \left[15-3\right] = -(3) - (12) = -15$$

3. Use row-reducing methods to find the determinant of $\begin{bmatrix} 1 & 0 & -1 & 0 & -1 \\ -1 & -1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 1 & 1 & 0 \\ -1 & 1 & 1 & -1 & 0 \end{bmatrix}$. (7 points)

$R_4+R_5 \to R_5$

$$\begin{vmatrix} 1 & 0 & -1 & 0 & -1 \\ -1 & -1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 1 & 1 & 0 \\ -1 & 2 & 2 & 0 & 0 \end{vmatrix} \Rightarrow \begin{vmatrix} 1 & 0 & -1 & -1 \\ -1 & -1 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ -1 & 2 & 2 & 0 \end{vmatrix}$$

$R_1+R_2 \to R_2$
$R_2+R_3 \to R_3$
$R_1+R_4 \to R_4$

$$\begin{vmatrix} 1 & 0 & -1 & -1 \\ 0 & -1 & -1 & -2 \\ 0 & -1 & 0 & -2 \\ 0 & 2 & 1 & -1 \end{vmatrix} \Rightarrow$$

$$\begin{vmatrix} -1 & -1 & -2 \\ -1 & 0 & -2 \\ 2 & 1 & -1 \end{vmatrix} \quad R_1+R_3 \to R_3 \quad \begin{vmatrix} -1 & -1 & -2 \\ -1 & 0 & -2 \\ 1 & 0 & -3 \end{vmatrix} \to \begin{vmatrix} -1 & -2 \\ 1 & -3 \end{vmatrix} = 3+2 = 5$$

4. Use properties of determinants and $\det(A) = -3, \det(B) = 2$, to find the following, given that both A and B are 5×5 matrices. (2 points each)

a. $\det(B^{-1})$ $\quad \frac{1}{2}$

b. $\det(A^{-1}B^2)$ $\quad -\frac{1}{3}(4) = -\frac{4}{3}$

c. $\det(4A)$ $\quad 4^5(-3) = -3072$

d. $\det(A^{-1}BA)$ $\quad -\frac{1}{3}(2)(-3) = 2$

e. $\det(-B^T)$ $\quad (-1)^5 2 = -2$

5. Determine if the following sets of vectors are independent. If do, do they form a basis for R^n? Explains your reasoning in each case. (3 points each)

a. $\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$

independent, spans $\mathbb{R}^2$ is a basis
(not multiples)

b. $\vec{v}_1 = \begin{bmatrix} 1 \\ -6 \\ 2 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 4 \\ 7 \\ 1 \end{bmatrix}$

not independent ($\vec{0}$)
does not span $\mathbb{R}^3$, not a basis

c. $\vec{v}_1 = \begin{bmatrix} 4 \\ 2 \\ -6 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -2 \\ -1 \\ 3 \end{bmatrix}$

$-2\vec{v}_2 = \vec{v}_1$ dependent
not a basis for $\mathbb{R}^3$

d. $\vec{v}_1 = \begin{bmatrix} 3 \\ -1 \\ -1 \\ 2 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 3 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \vec{v}_4 = \begin{bmatrix} -2 \\ -4 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_5 = \begin{bmatrix} 3 \\ -4 \\ 0 \\ 4 \end{bmatrix}$

not independent too many vectors to
be independent in $\mathbb{R}^4$; not a basis

6. What is the dimension of the space defined by the set of functions $S = \{e^x, e^{-x}, \cosh(2x), \cosh x\}$? [Note: $\sinh(t) = \frac{e^t - e^{-t}}{2}, \cosh(t) = \frac{e^t + e^{-t}}{2}$.] Explain your reasoning. (4 points)

dimension is 3 Since cosh x is dependent
on e^x & e^{-x} (a linear combination)

7. If $\det(A) = 4$, use properties of determinants to find the following: (2 points each)
   a. The determinant of the matrix B where B is the matrix formed from A by applying $R_1 + 2R_2 \to R_2$ to A.

      8

   b. The determinant of the matrix C where C is the matrix formed from A by applying $7R_4 + 3R_1 - 4R_5 \to R_5$ to A.

      -16

   c. The determinant of the matrix D where D is the matrix formed from A by applying $R_3 \leftrightarrow R_{10}$ to A.

      -4

8. Let S be a subset of P_5 and consist of all polynomials of the form $p(x) = ax^2 + 1$. Determine if S is a subspace. If it is, prove it. If it is not, find a counterexample. (5 points)

It is not

if $a=0$ $p(x)=1 \neq 0$

Since $\vec{0}$ does not exist in the set, it is not a subspace

9. Let M be a 7×8 matrix. If M has 5 pivots, find the following: (1 point each)

a. $\dim Col\ A$ 5

b. $rank\ A$ 5

c. $\dim Nul\ A$ 3

d. $nullity\ of\ A$ 3

e. $Col\ A$ is a subspace of R^m. $m = ?$ 7

f. $Nul\ A$ is a subspace of R^n. $n = ?$ 8

MAT 202, Exam #2, Part II, Summer 2017 Name ______________________

Instructions: Show all work. You **may** use a calculator on this portion of the exam. To show work on calculator problems, show the commands you used, and the resulting matrices. **Give exact answers** (yes, that means fractions, square roots and exponentials, and not decimals) unless specifically directed to give a decimal answer. This will require some operations to be done by hand even if not specifically directed to. Be sure to complete all parts of each question. Provide explanations where requested.

1. Use Cramer's Rule to solve the system $\begin{cases} 2x + 3y + 2z = -2 \\ -x - 3y - 8z = -2 \\ -3x + 2y - 7z = 2 \end{cases}$. Show all the required matrices. You may use your calculator to find the required determinants. State your final solution as a vector. (5 points)

$$\det A = \begin{vmatrix} 2 & 3 & 2 \\ -1 & -3 & -8 \\ -3 & 2 & -7 \end{vmatrix} = 103$$

$$x_1 = \frac{-160}{103}$$

$$\det A_1 = \begin{vmatrix} -2 & 3 & 2 \\ -2 & -3 & -8 \\ 2 & 2 & -7 \end{vmatrix} = -160$$

$$x_2 = \frac{10}{103}$$

$$x_3 = \frac{42}{103}$$

$$\det A_2 = \begin{vmatrix} 2 & -2 & 2 \\ -1 & -2 & -8 \\ -3 & 2 & -7 \end{vmatrix} = 10$$

$$\det A_3 = \begin{vmatrix} 2 & 3 & -2 \\ -1 & -3 & -2 \\ -3 & 2 & 2 \end{vmatrix} = 42$$

2. Write $\vec{v} = \begin{bmatrix} -3 \\ -3 \\ 1 \end{bmatrix}$ as a linear combination of $\vec{u}_1 = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} 0 \\ -1 \\ -2 \end{bmatrix}, \vec{u}_4 = \begin{bmatrix} -3 \\ -1 \\ -2 \end{bmatrix}$. If there are infinite number of ways, write one example along with the form of the dependent solution. (3 points)

$$\left[\begin{array}{cccc|c} -1 & 0 & 0 & -3 & -3 \\ -1 & -1 & -1 & -1 & -3 \\ 2 & -1 & -2 & -2 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 3 & 3 \\ 0 & 1 & 0 & -12 & -5 \\ 0 & 0 & 1 & 10 & 5 \end{array}\right]$$

$$3\vec{u} - 5\vec{u}_2 + 5\vec{u}_3 + 0\vec{u}_4 = \vec{v}$$

$$\vec{x} = \begin{bmatrix} -3 \\ \cdots \end{bmatrix} t + \begin{bmatrix} 3 \\ -5 \\ \cdots \end{bmatrix}$$

3. Write a system of equations to determine if $M = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$ can be written as a linear combination of $M_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, M_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, M_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. You don't need to solve the system. (3 points)

$$a\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$$

$a = 2$

$b = 1$

$-a + c = 4$

$0 = 3$

Can't be solved
no way to get $\begin{bmatrix} - & - \\ 3 & - \end{bmatrix}$
here

4. Determine if $p_1(x) = x^3 - 2x^2 + 1, p_2(x) = 5x, p_3(x) = x^2 - 4, p_4(x) = x^3 + 2x$, forms a basis for $\boldsymbol{P}_3$. If it is, write $q(x) = x^3 - 1$ as a linear combination of the basis vectors. If it is not, explain your reasoning. (3 points)

$$\underbrace{\begin{bmatrix} 1 & 0 & -4 & 0 \\ 0 & 5 & 0 & 2 \\ -2 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}}_{A} \left| \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right.$$

it is a basis since A maps to identity

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -3/7 \\ 0 & 1 & 0 & 0 & -4/7 \\ 0 & 0 & 1 & 0 & 1/7 \\ 0 & 0 & 0 & 1 & 10/7 \end{array}\right]$$

$-\frac{3}{7}p_1(x) + (-4/7)p_2(x) + \frac{1}{7}p_3(x) + \frac{10}{7}p_4(x) = q(x)$

5. Determine the dimensions of the following vector spaces or subspaces. (2 points each)
   a. $\boldsymbol{P}_7$

   8

   b. $\boldsymbol{C}^4$

   8

c. Symmetric 3×3 matrices

$$\begin{bmatrix} a & b & d \\ b & c & e \\ d & e & f \end{bmatrix} \quad 6$$

d. set of all continuous functions such that $f(0) = 0$.

∞

e. $S = \left\{ \begin{bmatrix} -s \\ s - 5t \\ 3t + 2s \end{bmatrix} \middle| s, t \in \boldsymbol{R} \right\}$

2

$$\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} s + \begin{bmatrix} 0 \\ -5 \\ 3 \end{bmatrix} t$$

6. If $\vec{v} = \begin{bmatrix} -2 \\ 2 \\ 2 \end{bmatrix}$ in the standard basis and $B = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$ and $C = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} \right\}$, find the following: (2 points each)

a. The transition matrix P_B

$$P_B = \begin{bmatrix} -2 & 1 & -1 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix}$$

b. The transition matrix $P_{C \leftarrow B}$

$$P_C^{-1} P_B = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & 2 & 0 \end{bmatrix}^{-1} \begin{bmatrix} -2 & 1 & -1 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 4/3 & -4/3 \\ 1 & 1/3 & 2/3 \\ 0 & 1/3 & -1/3 \end{bmatrix}$$

c. $[\vec{v}]_B$

$$P_B [\vec{x}]_B = \vec{x} \qquad P_B^{-1} \vec{x} = [\vec{x}]_B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

d. $[\vec{v}]_C$

$$P_C^{-1} \vec{x} = [\vec{x}]_C = \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}$$